

Preface

This volume is devoted to the foundations of the theory of moduli of algebraic curves defined over the complex numbers. The first volume was almost exclusively concerned with the geometry on a fixed, smooth curve. At the time it was published, the local deformation theory of a smooth curve was well understood, but the study of the geometry of global moduli was in its early stages. This study has since undergone explosive development and continues to do so. There are two reasons for this; one predictable at the time of the first volume, the other not.

The predictable one was the intrinsic algebro-geometric interest in the moduli of curves; this has certainly turned out to be the case. The other is the external influence from physics. Because of this confluence, the subject has developed in ways that are incredibly richer than could have been imagined at the time of writing of Volume I.

When this volume, GAC II, was planned it was envisioned that the centerpiece would be the study of linear series on a general or variable curve, culminating in a proof of the Petri conjecture. This is still an important part of the present volume, but it is not the central aspect. Rather, the main purpose of the book is to provide comprehensive and detailed foundations for the theory of the moduli of algebraic curves. In addition, we feel that a very important, perhaps distinguishing, aspect of GAC II is the blending of the multiple perspectives—algebro-geometric, complex-analytic, topological, and combinatorial—that are used for the study of the moduli of curves.

It is perhaps keeping this aspect in mind that one can understand our somewhat unusual choice of topics and of the order in which they are presented. For instance, some readers might be surprised to see a purely algebraic proof of the projectivity of moduli spaces immediately followed by a detailed introduction to Teichmüller theory. And yet Teichmüller theory is needed for our subsequent discussion of smooth Galois covers of moduli, which in turn is immediately put to use in our approach to the theory of cycles on moduli spaces. Besides, all the above are essential tools in Kontsevich's proof of Witten's conjecture, which is presented in later chapters. Concerning this, the main motivation of our choice of presenting Kontsevich's original proof instead of one of the several more recent ones is—in addition to the great beauty of the proof itself—a desire to be as self-contained as possible. This same desire also motivates in part the presence, at the beginning of the book,

of two introductory chapters on the Hilbert scheme and on deformation theory.

In the *Guide for the Reader* we will briefly go through the material we included in this volume. Among the topics we did not cover are the theory of Gromov–Witten invariants, the birational geometry of moduli spaces, the theory of moduli of vector bundles on a fixed curve, the theory of syzygies for the canonical curve, the various incarnations of the Schottky problem together with the related theory of theta function, and the theory of stable rational cohomology of moduli spaces of smooth curves. Some of these topics are covered by excellent publications like [14] for syzygies and [532] for the birational geometry of moduli spaces. On other topics, like the intersection theory of cycles or the theory of the ample cone of moduli spaces of stable curves, we limited ourselves to the foundational material.

Much of Volume I was devoted to the study of the relationship between an algebraic curve and its Jacobian variety. In this volume there is relatively little emphasis on the universal Jacobian or Picard variety and discussion of the moduli of abelian varieties. The latter is a vast and deep subject, especially in its arithmetic aspect, that goes well beyond the scope of this book.

In some instances, important topics, such as the Kodaira dimension of moduli spaces of stable curves, the theory of limit linear series, or the irreducibility of the Severi variety, have appeared elsewhere, specifically in the book *Moduli of Curves* by Joe Harris and Ian Morrison [352]. This is in fact a good opportunity to thank Joe and Ian for their kind words in the introduction of their book. We believe that our respective books complement each other, and we encourage our readers to benefit from their work.

In the bibliographical notes we try to point the reader to the most significant developments, not covered in this volume, of which we were aware at the time of writing. In fact, we view our bibliography and our bibliographical notes as, potentially, an ongoing project.

There is virtually no area in the theory of moduli of curves where the contribution of David Mumford has not been crucial. Our first debt of gratitude is therefore owed to him.

There is a long list of people to whom we would also like to express our gratitude. The first one is Joe Harris, whose generous contribution consists of approximately half of the exercises in this book.

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