Preface

The main theme of these notes is the study, from the standpoint of s-numbers, of operators of Hardy type and related Sobolev embeddings. More precisely, let $p,q \in (1,\infty)$ and suppose that I is the interval (a,b), where $-\infty < a < b < \infty$. Maps $T : L_p(I) \to L_q(I)$ of the form

$$(Tf)(x) = v(x)\int_a^x u(t)f(t)dt, \tag{1}$$

where u and v are prescribed functions satisfying some integrability conditions, are said to be of Hardy type. They are of importance in connection with 'small ball' problems in probability theory [87] and also in the theory of embeddings of Sobolev spaces when the underlying subset Ω of $\mathbb{R}^n$ is a generalised ridged domain, which means crudely that Ω has a central axis (the generalised ridge) that is the image of a tree under a Lipschitz map [42]. In addition, the literature on such maps T has grown to such an extent that the topic has acquired an independent life. Our object is, so far as we are able, to give an account of the present state of knowledge in this area in the hope that it will stimulate further work. In addition to the main theme, topics that arise naturally include the geometry of Banach spaces, generalised trigonometric functions and the p-Laplacian, and we have not hesitated to develop these subsidiary melodies beyond the strict requirements of Hardy operators when the intrinsic interest warranted it. We hope that the resulting contrapuntal effect will appeal to the reader.

Chapter 1 supplies basic information about bases of Banach spaces and such geometric concepts as strict and uniform convexity, uniform smoothness and super-reflexivity. It also gives an account of very recent work (see [44]) on the representation of compact linear operators $S : X \to Y$, where X and Y are reflexive Banach spaces with strictly convex duals. What emerges is the existence of a sequence (x_n) in the unit sphere of X and a sequence (λ_n) of positive numbers in terms of which the action of S can be described and points $x \in X$ represented, under suitable conditions; the λ_n are norms of the restrictions of S to certain subspaces. These results provide an analogue in Banach spaces of the celebrated Hilbert space results of Erhard Schmidt. As a byproduct we have (in Chap. 3) a proof of the existence of an infinite sequence of 'eigenvectors' of the Dirichlet problem for the p-Laplacian in an arbitrary bounded domain in $\mathbb{R}^n$.

The next chapter gives an account of generalised trigonometric functions. To explain what is involved here, let $p \in (1,\infty)$, put

$$\pi_p = \frac{2\pi}{p \sin(\pi/p)}$$

and let $F_p : [0,1] \to \mathbb{R}$ be given by

$$F_p(x) = \int_0^x (1-t^p)^{-1/p} dt.$$

Then the generalised sine function $\sin_p$ is the function defined on $[0, \pi_p/2]$ to be the inverse of F_p and extended to the whole of $\mathbb{R}$ in a natural way so as to be $2\pi_p$-periodic. Plainly $\sin_2 = \sin$. Moreover, p-analogues of the other trigonometric functions may easily be given: for example, $\cos_p$ is defined to be the derivative of $\sin_p$, from which it follows quickly that

$$\left|\sin_p x\right|^p + \left|\cos_p x\right|^p = 1 \text{ for all } x \in \mathbb{R}.$$

After establishing the main properties of these p-functions and some of the identities obtainable by their use, such as a new representation of the Catalan constant, the chapter finishes with a proof of the fact (first given in [9]) that if p is not too close to 1, then the functions $\sin_p(n\pi_p t)$ form a basis in $L_q(0,1)$ for all $q \in (1,\infty)$. The usefulness of such p-functions is underlined in Chap. 3, where it is shown how $\sin_p$ and $\cos_p$ arise naturally in the study of initial- and boundary-value problems for the one-dimensional p-Laplacian on an interval.

Chapter 4 provides necessary and sufficient conditions for the boundedness and compactness of the Hardy operator T of (1) acting between Lebesgue spaces. The norm of T_0, the particular form of T when $u = v = 1$, is determined explicitly and is shown to be attained at functions expressible in terms of generalised trigonometric functions. After this preparation, Chap. 5 is devoted to the s-numbers of T_0, together with the calculation of s-numbers of the basic Sobolev embedding on intervals. We remind the reader that in the theory of s-numbers, to every bounded linear map $S : X \to Y$, where X and Y are Banach spaces, is attached a non-increasing sequence $(s_n(S))_{n\in\mathbb{N}}$ of non-negative numbers with a view to classifying operators according to the behaviour of $s_n(S)$ as $n \to \infty$. The approximation numbers are particularly important examples: the nth approximation number of S is defined to be

$$a_n(S) = \inf \|S - F\|,$$

where the infimum is taken over all linear maps $F : X \to Y$ with rank less than n. These are special cases of the so-called "strict" s-numbers, further examples of which are provided by the Bernstein, Gelfand, Kolmogorov and Mityagin numbers. As might be expected, the results obtained regarding T_0 are especially sharp when $p = q$. In fact, it then turns out that all the strict s-numbers of T_0 coincide, the nth such number $s_n(T_0)$ being given by the formula

$$s_n(T_0) = \frac{(b-a)\gamma_p}{n+1/2}, \text{ where } \gamma_p = \frac{1}{2\pi} p^{1/p'} (p')^{1/p} \sin(\pi/p).$$

Chapter 6 deals with the general case of the operator T given by (1), in which u and v are merely required to satisfy certain integrability conditions. The precision of the results for T_0 is not obtainable for $T : L_p(I) \to L_p(I)$, but it emerges that if $1 < p < \infty$, then again all the strict s-numbers of T coincide, and that this time the asymptotic formula

$$\lim_{n\to\infty} ns_n(T) = \gamma_p \int_a^b |u(t)v(t)|\, dt$$

holds, where $s_n(T)$ denote the common value of the nth strict s-number of T. The cases $p = 1$ and ∞ present particular difficulties, but even then upper and lower estimates for the approximation numbers of T are obtained. The next chapter develops the theme of Chap. 6: it includes the derivation of more precise asymptotic information about the strict s-numbers of T, given additional restrictions on u and v.

So far, knowledge of the behaviour of the s-numbers of T has been obtained only for the case in which T acts from $L_p(I)$ to itself. When T is viewed as a map from $L_p(I)$ to $L_q(I)$ and $p \neq q$, special problems arise and new techniques are required. Chapter 8 deals with this situation and obtains results by consideration of the variational problem of determining

$$\sup_{g\in T(B)} \|g\|_q,$$

where B is the closed unit ball in $L_p(I)$. When $1 < q < p < \infty$, the asymptotic behaviour of the approximation numbers and the Kolmogorov numbers is established: thus

$$\lim_{n\to\infty} na_n(T) = C(p,q) \left(\int_a^b |u(t)v(t)|^r\, dt \right)^{1/r},$$

where $C(p,q)$ is an explicitly known function of p and q, and $r = 1/q + 1/p'$. Moreover, when $1 < p < q < \infty$, a corresponding formula is shown to hold for the Bernstein numbers of T. In both cases connections are made between the s-numbers of T and 'eigenvalues' of the variational problem mentioned above. We stress the key rôle played in the arguments presented in Chaps. 5–8 by the generalised trigonometric functions; Chap. 8 also uses more topological ideas, such as the Borsuk antipodal theorem.

The final chapter extends the discussion of the Hardy operator to the situation in which it acts on spaces with variable exponent, the $L_{p(\cdot)}$ spaces. Here p is a given function with values in $(1,\infty)$: if p is a constant function the space coincides with the usual L_p space. Such spaces have attracted a good deal of interest lately because they occur naturally in various physical contexts and in variational problems involving integrands with non-standard growth properties.