I. General notions on magic squares

A *magic square* is a square divided into a square number of cells in which natural numbers, all different, are arranged in such a way that the same sum is found in each horizontal row, each vertical row, and each of the two main diagonals.

A square with n cells on each side, thus n^2 cells altogether, is said to have the *order* n. The constant sum to be found in each row is called the *magic sum* of this square.

Usually what is written in a square of order n, thus with n^2 cells, are the first n^2 natural numbers. Since the sum of all these numbers equals

$$\frac{n^2\,(n^2+1)}{2},$$

the sum in each row, thus the magic sum for such a square, will be

$$M_n = \frac{n(n^2+1)}{2}.$$

I. A square displaying this magic sum in the $2n+2$ aforesaid rows is an *ordinary magic square* (Fig. 1). It meets the minimum number of required conditions, and such a square can be constructed for any given order $n \geq 3$ (a magic square of order 2 is not possible with different numbers).

1	**32**	**34**	**3**	**35**	**6**
30	**8**	**27**	**28**	**11**	**7**
19	**23**	**15**	**16**	**14**	**24**
18	**17**	**21**	**22**	**20**	**13**
12	**26**	**10**	**9**	**29**	**25**
31	**5**	**4**	**33**	**2**	**36**

Fig. 1

But there is an important, other kind.

II. A *bordered magic square* is one where removal of its successive borders leaves each time a magic square (Fig. 2). With an odd-order square, after removing each (odd-order) border in turn, we shall finally reach the smallest possible square, that of order 3. With an even-order square, that will be one of order 4 (its border cannot be removed since there is no magic square of order 2). For order $n \geq 5$, bordered squares are always possible.

16	*81*	*79*	*77*	*75*	**11**	**13**	**15**	**2**
78	**28**	*65*	*63*	*61*	**25**	**27**	**18**	**4**
76	*62*	**36**	*53*	*51*	**35**	**30**	**20**	**6**
74	*60*	*50*	**40**	*45*	**38**	**32**	**22**	**8**
9	**23**	**33**	**39**	**41**	*43*	*49*	*59*	*73*
10	**24**	**34**	*44*	**37**	*42*	*48*	*58*	*72*
12	**26**	*52*	**29**	**31**	*47*	*46*	*56*	*70*
14	*64*	**17**	**19**	**21**	*57*	*55*	*54*	*68*
80	**1**	**3**	**5**	**7**	*71*	*69*	*67*	*66*

Fig. 2

As seen above, the magic sum for a square of order n filled with the n^2 first natural numbers is

$$M_n = \frac{n(n^2+1)}{2}.$$

Clearly, the *average* sum in each case is $\frac{n^2+1}{2}$. Accordingly, for m cells, the average sum should be m times that quantity; this will be called the *sum due* for m cells. Thus, the inner square of order m ($m \geq 5$) within a bordered square of order n must contain in each row its sum due, namely

$$M_n^{(m)} = \frac{m(n^2+1)}{2}$$

if the main square is filled with the n^2 first natural numbers. The sum in a row in one border will therefore differ from the next by n^2+1. From this it follows that pairs of elements of the same border which are horizontally and vertically (diagonally for the corner cells) opposite add up to n^2+1. Such pairs are what we call *complements*: to each 'small' number a (less than $\frac{n^2+1}{2}$) is associated the 'large' number $(n^2+1)-a$. For example, in each border of the above figure the sum of opposite elements is 82, while 41 belongs to the centre and $1, \ldots, 40$ is the set of small numbers. We therefore infer that, for a bordered square of odd order filled with the first natural numbers, the central element must be $\frac{n^2+1}{2}$ (the *median*). The case of even-order squares is similar, except that there are two median numbers, $\frac{n^2}{2}$ and $\frac{n^2}{2}+1$, to be placed in the central 4×4 square, the small numbers being those less than (or equal to) $\frac{n^2}{2}$.