1

Basics of Continuum Mechanics

1.1 Vectors and Tensors

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Continuum mechanics relies on vectors and tensors. We first introduce vector/tensor operations. In this book, scalars are denoted by italic lowercase letters, while vectors and tensors are represented by upright boldface letters.

1.1.1 Vector

If a physical quantity contains only magnitude information, such as time, mass, temperature, density, and length, it is referred to as a scalar. A vector represents a directed line segment in space. The length of the line segment indicates the magnitude of the vector, while the direction of the line segment represents the direction of the vector. It is used to describe physical quantities that have both direction and magnitude, such as displacement, velocity, acceleration, and force.

The addition of vectors follows the same rules as scalar addition, satisfying the commutative law and the associative law of addition:

$$\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}, \tag{1.1a}$$

$$(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w}). \tag{1.1b}$$

The sum of two vectors is still a vector, and the magnitude and direction of the resultant vector are determined according to the parallelogram law.

The product of a vector $\mathbf{u}$ and a scalar α is a new vector that has the same direction as $\mathbf{u}$ if $\alpha > 0$, or the opposite direction if $\alpha < 0$. The multiplication of a scalar and a vector satisfies the commutative law, the associative law, and the distributive law as follows:

$$\alpha\mathbf{u}=\mathbf{u}\alpha, \tag{1.2a}$$

$$(\alpha\beta)\mathbf{u}=\alpha(\beta\mathbf{u}), \tag{1.2b}$$

$$(\alpha+\beta)\mathbf{u}=\alpha\mathbf{u}+\beta\mathbf{u}, \tag{1.2c}$$

$$\alpha(\mathbf{u}+\mathbf{v})=\alpha\mathbf{u}+\alpha\mathbf{v}. \tag{1.2d}$$

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The dot product of vectors $\mathbf{u}$ and $\mathbf{v}$, denoted by $\mathbf{u} \cdot \mathbf{v}$, results in a scalar:

$$\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}| \cos\theta, \tag{1.3}$$

where θ is the angle between two vectors $\mathbf{u}$ and $\mathbf{v}$. The dot product of vectors has the following properties:

- $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$,
- $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$.

The cross product of vectors $\mathbf{u}$ and $\mathbf{v}$, denoted by $\mathbf{u} \times \mathbf{v}$, results in a vector. Its magnitude is given by:

$$|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}||\mathbf{v}| \sin\theta. \tag{1.4}$$

The direction of $\mathbf{u} \times \mathbf{v}$ is perpendicular to the plane formed by $\mathbf{u}$ and $\mathbf{v}$, which follows the right-hand rule. The cross product of vectors has the following properties:

- $\mathbf{u} \times \mathbf{v} = -\mathbf{v} \times \mathbf{u}$,
- $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w}$.

1.1.2 Index Notation

To introduce the coordinate (or component) expression of a vector in a right-handed orthogonal system, we first introduce a set of fixed basis vectors $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$, called a Cartesian basis, with the following properties:

$$\mathbf{e}_1 \cdot \mathbf{e}_2 = \mathbf{e}_1 \cdot \mathbf{e}_3 = \mathbf{e}_2 \cdot \mathbf{e}_3 = 0, \quad \mathbf{e}_1 \cdot \mathbf{e}_1 = \mathbf{e}_2 \cdot \mathbf{e}_2 = \mathbf{e}_3 \cdot \mathbf{e}_3 = 1. \tag{1.5}$$

These three vectors are of unit length and mutually orthogonal. A vector in three-dimensional space can be expressed as

$$\mathbf{u} = u_1\mathbf{e}_1 + u_2\mathbf{e}_2 + u_3\mathbf{e}_3, \tag{1.6}$$

which can be written in summation notation as

$$\mathbf{u} = \sum_{i=1}^{3} u_i\mathbf{e}_i. \tag{1.7}$$

This expression can be further simplified by using a dummy index to omit the summation symbol, written as

$$\mathbf{u} = u_i\mathbf{e}_i. \tag{1.8}$$

An index that repeats exactly twice in an expression is called a dummy index, which signifies the summation convention. An index that is not summed over in an expression is referred to as a free index. For example:

$$f_i = u_i v_j w_j, \tag{1.9a}$$

$$f_1 = u_1(v_1w_1 + v_2w_2 + v_3w_3), \tag{1.9b}$$

$$f_2 = u_2(v_1w_1 + v_2w_2 + v_3w_3), \tag{1.9c}$$

$$f_3 = u_3(v_1w_1 + v_2w_2 + v_3w_3), \tag{1.9d}$$

where i is the free index and j is the dummy index.

The calculus operations involving scalars and vectors often carry clear physical significance. We begin by introducing two commonly used symbols in continuum mechanics: the Kronecker delta δ_{ij} and the permutation symbol ε_{ijk}. The Kronecker delta δ_{ij} is defined as follows:

$$\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij} \equiv \begin{cases} 1, & \text{if} \quad i = j, \\ 0, & \text{if} \quad i \neq j. \end{cases}$$

The following are some of its useful properties:

$$\delta_{ii} = 3, \quad \delta_{ij}u_i = u_j, \quad \delta_{ij}\delta_{jk} = \delta_{ik}. \tag{1.10}$$

Another commonly used symbol is the permutation symbol ε_{ijk}, which is defined as follows:

$$\varepsilon_{ijk} = \begin{cases} 1, & \text{for even permutations of } (i, j, k) \text{ (i.e. } 123, 231, 312\text{).} \\ -1, & \text{for odd permutations of } (i, j, k) \text{ (i.e. } 132, 213, 321\text{).} \\ 0, & \text{if there is a repeated index.} \end{cases} \tag{1.11}$$

The permutation symbol has the following properties:

$$\varepsilon_{ijk} = \varepsilon_{jki} = \varepsilon_{kij}, \quad \varepsilon_{ijk} = -\varepsilon_{ikj}, \quad \varepsilon_{ijk} = -\varepsilon_{jik}. \tag{1.12}$$

1.1.3 Tensor

A second-order tensor can be viewed as a linear operator acting on a vector $\mathbf{u}$, producing a new vector $\mathbf{v}$ as

$$\mathbf{v} = \mathbf{A}\mathbf{u}. \tag{1.13}$$

A second-order tensor can be written in the index notation as

$$\mathbf{A} = A_{ij}\mathbf{e}_i \otimes \mathbf{e}_j. \tag{1.14}$$

where $\otimes$ is the dyad of vectors.

A unit tensor $\mathbf{I}$ can be represented using the Kronecker delta symbol as

$$\mathbf{I} = \delta_{ij}\mathbf{e}_i \otimes \mathbf{e}_j = \mathbf{e}_i \otimes \mathbf{e}_i. \tag{1.15}$$

The dot product of two second-order tensors $\mathbf{A}$ and $\mathbf{B}$ is typically denoted as $\mathbf{AB}$, which is still a second-order tensor. The components of the dot product $\mathbf{AB}$ can be represented as

$$(\mathbf{AB})_{ij} = A_{ik}B_{kj}. \tag{1.16}$$

The double dot product of two second-order tensors results in a scalar, as given by the following expression:

$$\mathbf{A} : \mathbf{B} = A_{ij}\mathbf{e}_i \otimes \mathbf{e}_j : B_{kl}\mathbf{e}_k \otimes \mathbf{e}_l = A_{ij}B_{kl}(\mathbf{e}_i \cdot \mathbf{e}_k)(\mathbf{e}_j \cdot \mathbf{e}_l) = A_{ij}B_{ij}. \tag{1.17}$$

The transpose of a second-order tensor $\mathbf{A}$ is denoted as $\mathbf{A}^{\mathrm{T}}$, with the following property:

$$(A^{\mathrm{T}})_{ij} = A_{ji}. \tag{1.18}$$

The trace of a second-order tensor $\mathbf{A}$ equals to the sum of the diagonal terms represented as

$$\mathrm{tr}\,(\mathbf{A}) = A_{ii}. \tag{1.19}$$

The determinant of a tensor $\mathbf{A}$ is the determinant of the matrix $\mathbf{A}$, represented as

$$\det(\mathbf{A}) = \varepsilon_{ijk} A_{1i} A_{2j} A_{3k}. \tag{1.20}$$

For a nonsingular tensor, namely, $\det(\mathbf{A}) \neq 0$, there exists a unique inverse of the tensor, denoted as $\mathbf{A}^{-1}$, which satisfies as

$$\mathbf{A}\mathbf{A}^{-1} = \mathbf{I}. \tag{1.21}$$

A tensor can be decomposed into a symmetric part and a skew part as

$$\mathrm{sym}(\mathbf{A}) = \frac{\mathbf{A} + \mathbf{A}^{\mathrm{T}}}{2}, \quad \mathrm{skew}(\mathbf{A}) = \frac{\mathbf{A} - \mathbf{A}^{\mathrm{T}}}{2}. \tag{1.22}$$

A second-order tensor has different component forms in different coordinate systems:

$$\mathbf{A} = A_{ij} \mathbf{e}_i \otimes \mathbf{e}_j \quad \text{or} \quad \mathbf{A} = \tilde{A}_{ij} \tilde{\mathbf{e}}_i \otimes \tilde{\mathbf{e}}_j. \tag{1.23}$$

These two component forms can be related through the transformation of the basis vectors corresponding to the two coordinate systems, as

$$\mathbf{e}_i = Q_{ij} \tilde{\mathbf{e}}_j \quad Q_{ij} = \mathbf{e}_i \cdot \tilde{\mathbf{e}}_j. \tag{1.24}$$

where Q_{ij} represents the direction cosines between the basis vectors of the two coordinate systems. $\mathbf{Q}$ is an orthogonal second-order matrix with $\mathbf{Q}^{\mathrm{T}} = \mathbf{Q}^{-1}$.

Furthermore, we have:

$$\tilde{A}_{ij} = Q_{ki} A_{kl} Q_{lj}, \quad [\tilde{\mathbf{A}}] = [\mathbf{Q}]^{\mathrm{T}} [\mathbf{A}][\mathbf{Q}]. \tag{1.25}$$

A vector can be treated as a tensor with order 1. Thus, higher order tensors can also be defined in a similar way. For example, tensors with an order 3 or 4 can be defined as

$$\mathcal{A} = A_{ijk} \mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k. \tag{1.26a}$$

$$\mathbb{A} = A_{ijkl} \mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_l. \tag{1.26b}$$

1.1.4 Gradient, Divergence, and Curl

We introduce the vector operator ∇, commonly referred to as the Nabla operator, which is frequently used in computations:

$$\nabla(\bullet) = \frac{\partial(\bullet)}{\partial x_i} \mathbf{e}_i = \frac{\partial(\bullet)}{\partial x_1} \mathbf{e}_1 + \frac{\partial(\bullet)}{\partial x_2} \mathbf{e}_2 + \frac{\partial(\bullet)}{\partial x_3} \mathbf{e}_3. \tag{1.27}$$

Thus, the gradient vector of a scalar field at any point is given as

$$\text{grad}\Phi = \nabla\Phi = \frac{\partial \Phi}{\partial x_i}\mathbf{e}_i. \tag{1.28}$$

The gradient of a vector is a second-order tensor, defined as follows:

$$\text{grad}\mathbf{u} = \nabla\mathbf{u} = \frac{\partial u_i}{\partial x_j}\mathbf{e}_i \otimes \mathbf{e}_j. \tag{1.29}$$

Similarly, the gradient operator can also be applied to a tensor. For example, the gradient of a second-order tensor **A** can be written as

$$\text{grad}\mathbf{A} = \nabla\mathbf{A} = \frac{\partial A_{ij}}{\partial x_k}\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k. \tag{1.30}$$

The Nabla operator can also be combined with a vector through dot and cross products. The dot product of the Nabla operator with a vector is referred to as the divergence of the vector:

$$\text{div}\mathbf{u} = \nabla \cdot \mathbf{u} = \frac{\partial u_j}{\partial x_i}\mathbf{e}_j \cdot \mathbf{e}_i = \frac{\partial u_j}{\partial x_i}\delta_{ij} = \frac{\partial u_i}{\partial x_i}. \tag{1.31}$$

The divergence of a second-order tensor is represented as

$$\text{div}\mathbf{A} = \nabla \cdot \mathbf{A} = \frac{\partial A_{ij}}{\partial x_k}\mathbf{e}_i \otimes \mathbf{e}_j \cdot \mathbf{e}_k = \frac{\partial A_{ij}}{\partial x_k}\mathbf{e}_i \delta_{jk} = \frac{\partial A_{ij}}{\partial x_j}\mathbf{e}_i. \tag{1.32}$$

If the Nabla operator is dotted with the gradient of a scalar field, the result is

$$\nabla \cdot \nabla\Phi = (\nabla \cdot \nabla)\Phi = \nabla^2\Phi = \frac{\partial^2 \Phi}{\partial x_1^2} + \frac{\partial^2 \Phi}{\partial x_2^2} + \frac{\partial^2 \Phi}{\partial x_3^2}. \tag{1.33}$$

The ∇^2 operator is known as the Laplacian operator.

The cross product of the Nabla operator with a vector is referred to as the curl of the vector:

$$\text{curl}\mathbf{u} = \nabla \times \mathbf{u} = \frac{\partial u_j}{\partial x_i}\mathbf{e}_i \times \mathbf{e}_j = \varepsilon_{ijk}\frac{\partial u_j}{\partial x_i}\mathbf{e}_k, \tag{1.34}$$

which indicates the curl of a vector field representing the degree of rotation of the field vectors around any point.

1.2 Kinematics

1.2.1 Deformation Gradient

Different from the small strain condition, finite deformation needs to consider higher order geometric and mechanical quantities. The analysis based on the undeformed or deformed configuration leads to different descriptions of the same deformation: material (Lagrangian) description or spatial (Eulerian) description.

In the theory of continuum mechanics, a body is assumed to possess continuity in both space and time (or at least be piecewise continuous), allowing its various physical quantities to be represented by continuous functions. As illustrated in Figure 1.1, the continuous

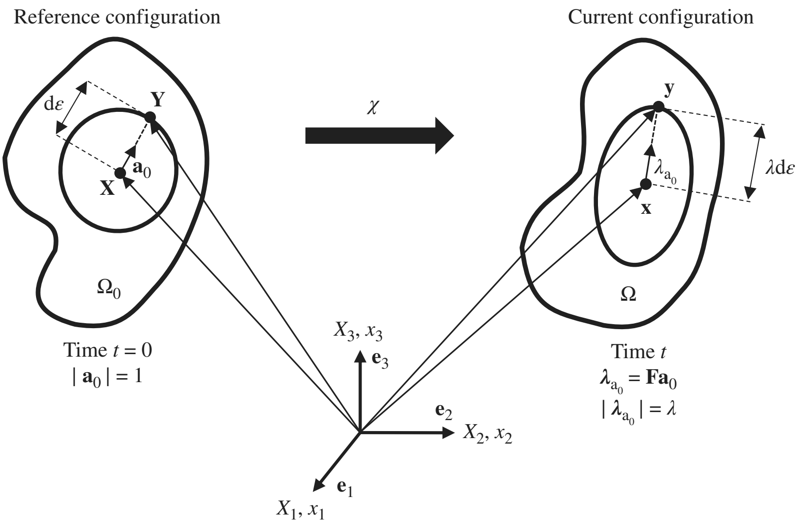

Figure 1.1 Deformation of a continuous medium.

spatial regions it occupies before and after a time interval are denoted by Ω_0 and Ω, respectively. The position of any point between these two configurations is related by a mapping:

$$\mathbf{x} = \chi(\mathbf{X}, t), \quad \mathbf{X} = \chi^{-1}(\mathbf{x}, t), \tag{1.35}$$

where $\mathbf{x}$ represents the position of any point in the current configuration and $\mathbf{X}$ represents the corresponding position of that point in the reference configuration.

With the coordinates of each point in both the reference configuration and the current configuration, the displacement of each point can be computed as follows:

$$\mathbf{u} = \mathbf{x} - \mathbf{X}. \tag{1.36}$$

The deformation gradient $\mathbf{F}$ is then introduced to relate the point vector in the reference configuration and that in the current configuration, which is represented as

$$\mathbf{F} = \frac{\partial \mathbf{x}}{\partial \mathbf{X}}, \quad F_{aA} = \frac{\partial x_a}{\partial X_A}. \tag{1.37}$$

We can further derive the relationship between the deformation gradient tensor and the displacement field as follows:

$$\mathbf{F} = \mathbf{I} + \frac{\partial \mathbf{u}}{\partial \mathbf{X}}, \quad F_{aA} = \delta_{aA} + \frac{\partial u_a}{\partial X_A}. \tag{1.38}$$

We can then easily relate the vector line in the reference and current configurations as

$$\mathrm{d}\mathbf{x} = \mathbf{F}\mathrm{d}\mathbf{X}. \tag{1.39}$$

The determinant of the deformation gradient matrix det$\mathbf{F}$ (or J) represents the deformation state of the infinitesimal volume element as

$$\mathrm{d}v = J\mathrm{d}V, \tag{1.40}$$

where $\mathrm{d}V$ and $\mathrm{d}v$ are the infinitesimal volume elements defined in the undeformed and deformed configurations, respectively. When $\det\mathbf{F} = 1$, the deformation is isochoric, meaning the infinitesimal volume element remains constant.

The infinitesimal volume element can be calculated through the dot product of a small surface and a material line element as

$$\mathrm{d}V = \mathrm{d}\mathbf{S} \cdot \mathrm{d}\mathbf{X}, \tag{1.41a}$$

$$\mathrm{d}v = \mathrm{d}\mathbf{s} \cdot \mathrm{d}\mathbf{x}, \tag{1.41b}$$

where $\mathrm{d}\mathbf{S}$ and $\mathrm{d}\mathbf{s}$ are the infinitesimally small areas in the reference and current configurations, respectively, and $\mathrm{d}\mathbf{X}$ and $\mathrm{d}\mathbf{x}$ are the line elements in the reference and current configurations, respectively.

Considering Eqs. (1.39)–(1.41), we can obtain:

$$\mathrm{d}\mathbf{s} = J\mathbf{F}^{-\mathrm{T}}\mathrm{d}\mathbf{S}. \tag{1.42}$$

This equation is known as Nanson's formula.

The deformation gradient contains both the contribution of rotation and stretch. To decompose the two effects, the deformation gradient $\mathbf{F}$ can be decomposed into

$$\mathbf{F} = \mathbf{R}\mathbf{U} = \mathbf{v}\mathbf{R}, \tag{1.43}$$

where $\mathbf{R}$ is an orthogonal rotation tensor, and $\mathbf{U}$ and $\mathbf{v}$ correspond to the right and left stretch tensors, respectively.

The time derivative of the deformation gradient $\mathbf{F}$ can be represented as

$$\dot{\mathbf{F}} = \frac{\partial}{\partial t}\left(\frac{\partial \mathbf{x}(\mathbf{X}, t)}{\partial \mathbf{X}}\right) = \frac{\partial}{\partial \mathbf{X}}\left(\frac{\partial \mathbf{x}(\mathbf{X}, t)}{\partial t}\right) = \frac{\partial \mathbf{V}}{\partial \mathbf{X}} = \mathrm{Grad}\mathbf{V}, \tag{1.44}$$

where $\mathbf{V}$ is the velocity defined in the reference configuration, and $\mathrm{Grad}(\cdot)$ is the gradient operator defined in the reference configuration.

Note: Different from the small strain condition, the operations, such as the gradient and divergence, need to be specified in the reference or current configurations. The relationship between these operations can be built using the chain rule [1].

The spatial velocity gradient can be further defined as

$$\mathbf{L} = \frac{\partial \mathbf{v}}{\partial \mathbf{x}} = \frac{\partial \mathbf{v}}{\partial \mathbf{X}}\frac{\partial \mathbf{X}}{\partial \mathbf{x}} = \frac{\partial}{\partial t}\left(\frac{\partial \mathbf{x}(\mathbf{X}, t)}{\partial \mathbf{X}}\right)\frac{\partial \mathbf{X}}{\partial \mathbf{x}} = \dot{\mathbf{F}}\mathbf{F}^{-1}. \tag{1.45}$$

1.2.2 Strain Tensor

We further measure the change in the squared distance between any material point and its neighboring point, represented as

$$\begin{aligned} (\mathrm{d}l)^2 - (\mathrm{d}L)^2 &= \mathrm{d}\mathbf{x} \cdot \mathrm{d}\mathbf{x} - \mathrm{d}\mathbf{X} \cdot \mathrm{d}\mathbf{X} \\ &= \mathrm{d}\mathbf{X} \cdot \mathbf{F}^{\mathrm{T}} \cdot \mathbf{F} \cdot \mathrm{d}\mathbf{X} - \mathrm{d}\mathbf{X} \cdot \mathrm{d}\mathbf{X} \\ &= \mathrm{d}\mathbf{X} \cdot (\mathbf{F}^{\mathrm{T}} \cdot \mathbf{F} - \mathbf{I}) \cdot \mathrm{d}\mathbf{X}, \end{aligned} \tag{1.46}$$

where $\mathrm{d}L$ and $\mathrm{d}l$ are the length in the reference and deformed configurations, respectively.

The previous equation provides the derivation of the Green–Lagrange strain tensor, which can be expressed as

$$\mathbf{E} = \frac{1}{2}(\mathbf{F}^{\mathrm{T}} \cdot \mathbf{F} - \mathbf{I}), \tag{1.47a}$$

$$E_{AB} = \frac{1}{2}(F_{aA}F_{aB} - \delta_{AB}). \tag{1.47b}$$

The square of the right stretch tensor and the left stretch tensor are, respectively, known as the right Cauchy–Green tensor **C** and left Cauchy–Green tensor **b**:

$$\mathbf{C} = \mathbf{U}^2 = \mathbf{F}^{\mathrm{T}}\mathbf{F}, \quad \mathbf{b} = \mathbf{v}^2 = \mathbf{F}\mathbf{F}^{\mathrm{T}}, \tag{1.48a}$$

$$\mathrm{C}_{AB} = F_{aA}F_{aB}, \quad b_{ab} = F_{aA}F_{bA}. \tag{1.48b}$$

Another common strain tensor is Euler–Almansi strain tensor **e**, defined as

$$\mathbf{e} = \frac{1}{2}(\mathbf{I} - \mathbf{F}^{-\mathrm{T}} \cdot \mathbf{F}^{-1}), \tag{1.49a}$$

$$e_{ab} = \frac{1}{2}(\delta_{ab} - F^{-1}_{Ca}F^{-1}_{Cb}). \tag{1.49b}$$

The right and left Cauchy–Green tensors share the same eigenvalues. The three invariants I_1, I_2, I_3 can be represented as

$$I_1 = C_{ii}, \tag{1.50a}$$

$$I_2 = \frac{1}{2}(C_{ii}C_{jj} - C_{ij}C_{ij}), \tag{1.50b}$$

$$I_3 = \det(\mathbf{C}). \tag{1.50c}$$

If the three principal stretches of the deformation gradient **F** are denoted as λ_1, λ_2, and λ_3, the three invariants can then be represented as

$$I_1 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2, \tag{1.51a}$$

$$I_2 = \lambda_1^2\lambda_2^2 + \lambda_2^2\lambda_3^2 + \lambda_1^2\lambda_3^2, \tag{1.51b}$$

$$I_3 = \lambda_1^2\lambda_2^2\lambda_3^2. \tag{1.51c}$$

1.3 Stress

The force exerted per unit area on an object is referred to as stress. To evaluate the stress state at a specific point within an object, we can hypothetically cut the object with a plane passing through that point, allowing us to obtain the stress at the point on the plane, which includes both magnitude and direction. Since there are infinitely many planes that can pass through a given point, it is impractical to describe the stress state at that point using an infinite number of stress vectors. This necessitates the introduction of the second-order stress tensor. The stress tensor effectively includes the information of the magnitude of the stress at a point, along with two directional components: one corresponding to the normal of the cutting plane and the other representing the projection direction of the stress tensor on that plane.

For an arbitrary plane, the force per unit area acting on this plane in the normal direction **n** is denoted as the Cauchy traction vector **t**. The Cauchy traction vector is then related to the Cauchy stress as

$$\mathbf{t} = \boldsymbol{\sigma}\mathbf{n}. \tag{1.52}$$

The Cauchy stress $\boldsymbol{\sigma}$ is a second-order symmetric stress tensor defined in the current configuration and is also referred to as the true stress tensor.

Correspondingly, we can also define the corresponding traction vector **T** in the reference configuration. The first Piola–Kirchhoff stress tensor **P** is related to the traction **T** as

$$\mathbf{T} = \mathbf{P}\mathbf{N}, \tag{1.53}$$

where **N** is the normal direction of the plane in the reference configuration.

The total force applied on the plane should be the same regardless of the configurations, which can be represented as

$$\mathbf{t}\mathrm{d}s = \mathbf{T}\mathrm{d}S, \tag{1.54}$$

where ds and dS are the areas in the current configuration and the reference configuration, respectively.

Combining the preceding three equations yields

$$\boldsymbol{\sigma}(\mathbf{x}, t)\mathbf{n}\mathrm{d}s = \mathbf{P}(\mathbf{X}, t)\mathbf{N}\mathrm{d}S, \tag{1.55a}$$

$$\boldsymbol{\sigma}(\mathbf{x}, t)\mathrm{d}\mathbf{s} = \mathbf{P}(\mathbf{X}, t)\mathrm{d}\mathbf{S}. \tag{1.55b}$$

Using Nanson's equation in Eq. (1.42), we can obtain the relationship between the first Piola–Kirchhoff stress and the Cauchy stress as

$$\mathbf{P} = J\boldsymbol{\sigma}\mathbf{F}^{-\mathrm{T}}. \tag{1.56}$$

Since the deformation gradient tensor **F** is not a symmetric tensor in general, **P** is also an asymmetric tensor in most cases. The second Piola–Kirchhoff stress tensor **S**, as a symmetric tensor, is further defined as

$$\mathbf{S} = \mathbf{F}^{-1}\mathbf{P} = J\mathbf{F}^{-1}\boldsymbol{\sigma}\mathbf{F}^{-\mathrm{T}}. \tag{1.57}$$

1.4 Balance Principles

Classic balance principles, such as conservation of mass, balance of momentum, and balance of energy, should also be satisfied in continuum mechanics. By applying the second law of thermodynamics to continuum mechanics, it can be expressed as an inequality governing the rate of change of free energy, thus imposing mathematical constraints on material deformation. In this chapter, we will introduce these balance principles together with the entropy inequality. It should be noted that the presentation in this book omits many details. For a complete derivative process, the readers can refer to classic continuum mechanics monographs [1–4].

1.4.1 Material Derivative and Spatial Derivative

The spatial derivative is defined as the time derivative of a physical or geometric quantity, with the current position $\mathbf{x}$ fixed, which can be expressed as

$$\frac{\partial \Phi}{\partial t} = \frac{\partial}{\partial t}\Phi(\mathbf{x}, t)\Big|_{\mathbf{x}}. \tag{1.58}$$

The material derivative is defined as the time derivative of a physical or geometric quantity, with the referential position $\mathbf{X}$ fixed, which can be expressed as:

$$\frac{\mathrm{d}\Phi}{\mathrm{d}t} = \frac{\partial}{\partial t}\Phi(\mathbf{x}, t)\Big|_{\mathbf{x}} + \frac{\partial \Phi}{\partial \mathbf{x}}\frac{\partial \mathbf{x}}{\partial t} = \frac{\partial}{\partial t}\Phi(\mathbf{x}, t)\Big|_{\mathbf{x}} + \mathrm{grad}\Phi \cdot \mathbf{v}. \tag{1.59}$$

1.4.2 Reynolds Transport Theorem

Suppose a body occupies a spatial domain Ω in three-dimensional space, with a volume v. Inside this domain, there exists a scalar field $\Phi = \Phi(\mathbf{x}, t)$. The time derivative of the integral of Φ over this domain cannot be directly evaluated by moving the derivative operation inside the integration function due to a continuous change of the volume element v. To compute this time derivative, we can use the following procedures:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega} \Phi(\mathbf{x}, t)\mathrm{d}v &= \frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega} \Phi(\mathbf{x}(\mathbf{X}, t), t)J\mathrm{d}V \\
&= \int_{\Omega_0} \left(J\frac{\mathrm{d}\Phi(\mathbf{x}(\mathbf{X}, t), t)}{\mathrm{d}t} + \Phi(\mathbf{x}(\mathbf{X}, t), t)\dot{J}\right)\mathrm{d}V \\
&= \int_{\Omega} \left(\frac{\mathrm{d}\Phi(\mathbf{x}(\mathbf{X}, t), t)}{\mathrm{d}t} + \Phi(\mathbf{x}(\mathbf{X}, t), t)\mathrm{div}\mathbf{v}\right)\mathrm{d}v.
\end{aligned} \tag{1.60}$$

In this derivative, we have used the relationship that $\dot{J}/J = \mathrm{div}\mathbf{v}$. This function is known as Reynolds Transport Theorem.

Using the relation in Eq. (1.59), Eq. (1.60) can be further rewritten as

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega} \Phi(\mathbf{x}, t)\mathrm{d}v &= \int_{\Omega} \left(\frac{\partial \Phi}{\partial t} + \mathrm{grad}\Phi \cdot \mathbf{v} + \Phi\mathrm{div}\mathbf{v}\right)\mathrm{d}v \\
&= \int_{\Omega} \left(\frac{\partial \Phi}{\partial t} + \mathrm{div}(\Phi\mathbf{v})\right)\mathrm{d}v \\
&= \int_{\Omega} \frac{\partial \Phi}{\partial t}\mathrm{d}v + \int_{\partial\Omega} \Phi\mathbf{v} \cdot \mathbf{n}\mathrm{d}s.
\end{aligned} \tag{1.61}$$

This equation is another form of Reynolds Transport Theorem. It shows the rate of change of the field Φ has two sources. The first part arises from the local rate of the field Φ within the region Ω. The second part is caused by the moving region.

1.4.3 Conservation of Mass

In a closed system, the total mass of material remains constant throughout motion and deformation for any chosen configuration. Mathematically, this can be expressed as the equivalence of integral of mass density over the volume in reference configuration and integral of mass density over the volume in current configuration:

$$\int_{\Omega_0} \rho_0(\mathbf{X})\mathrm{d}V = \int_{\Omega} \rho(\mathbf{x})\mathrm{d}v, \tag{1.62}$$

where ρ_0 is the mass density in reference configuration Ω_0 and ρ is the mass density in current configuration Ω. When this conclusion holds for an arbitrarily small neighborhood around each material point, it implies that the mass is locally conserved. This local conservation requirement can be expressed as

$$\rho_0 = J\rho. \tag{1.63}$$

Since $\dot{\rho}_0 = 0$, this gives $\dot{\overline{(J\rho)}} = 0$. This equation can be further rewritten as

$$\dot{\rho} + \rho \mathrm{div}\mathbf{v} = 0, \tag{1.64a}$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0. \tag{1.64b}$$

1.4.4 Balance of Momentum

Suppose a body with mass density ρ occupies a region Ω in its current configuration, with boundary $\partial\Omega$. The external force acting on the body can be divided into two components: one part is the body force $\mathbf{b}$ per unit volume within the region Ω, and the other part is the surface force $\mathbf{t}$ per unit area on the boundary $\partial\Omega$. The change rate of the momentum should be equal to the total applied force, represented as

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega} \rho \mathbf{v}\mathrm{d}v = \int_{\Omega} \mathbf{b}\mathrm{d}v + \oint_{\partial\Omega} \mathbf{t}\mathrm{d}s. \tag{1.65}$$

The surface integral of force can be expressed as

$$\oint_{\partial\Omega} \mathbf{t}\mathrm{d}s = \oint_{\partial\Omega} \boldsymbol{\sigma}\mathbf{n}\mathrm{d}s = \int_{\Omega} \nabla \cdot \boldsymbol{\sigma}\mathrm{d}v. \tag{1.66}$$

Using Reynolds Transport Theorem and the mass conservation law, we can then obtain the following expression [1]:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega} \rho \mathbf{v}\mathrm{d}v = \int_{\Omega} \rho \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t}\mathrm{d}v. \tag{1.67}$$

The law of conservation of momentum can then be written as

$$\int_{\Omega} (\nabla \cdot \boldsymbol{\sigma} + \mathbf{b}) \mathrm{d}v = \int_{\Omega} \rho \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} \mathrm{d}v. \tag{1.68}$$

Since the size of the region Ω is arbitrary, the previous expression can be written as the local form of the momentum conservation equation in the current configuration:

$$\nabla \cdot \boldsymbol{\sigma} + \mathbf{b} = \rho \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t}. \tag{1.69}$$

Under the condition of neglecting inertial term, the preceding equation degenerates to the equilibrium equation in elasticity:

$$\nabla \cdot \boldsymbol{\sigma} + \mathbf{b} = \mathbf{0}. \tag{1.70}$$

Using a similar process, the local form of the momentum conservation equation in the reference configuration can be obtained as

$$\nabla_{\mathrm{X}} \cdot \mathbf{P} + \mathbf{B} = \rho_0 \frac{\mathrm{d}\mathbf{V}}{\mathrm{d}t}, \tag{1.71}$$

where $\mathbf{B}$ is the body force in reference configuration.

1.4.5 Balance of Angular Momentum

The conservation of angular momentum states that the rate of change of the angular momentum about a certain point is equal to the resultant moment of external forces at that point. Consider a body with mass density ρ occupies a region Ω in its current configuration, with boundary $\partial\Omega$. The external moment can be divided into two components: one part is caused by the body force per unit volume within the region Ω, and the other part is induced by the surface force per unit area on the boundary $\partial\Omega$. Let $\mathbf{r}$ be the vector from the coordinate origin to a point on the body. Therefore, the balance of angular momentum in the current configuration can be expressed as

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega} \mathbf{r} \times \rho \mathbf{v} \mathrm{d}v = \int_{\Omega} \mathbf{r} \times \mathbf{b} \mathrm{d}v + \oint_{\partial\Omega} \mathbf{r} \times \mathbf{t} \mathrm{d}s. \tag{1.72}$$

According to Reynolds Transport Theorem, the left-hand side of the equation can be simplified as

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega} \mathbf{r} \times \rho \mathbf{v} \mathrm{d}v = \int_{\Omega} \rho \mathbf{r} \times \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} \mathrm{d}v. \tag{1.73}$$

The moment of the resultant external forces can be simplified to:

$$\int_{\Omega} \mathbf{r} \times \mathbf{b} \mathrm{d}v + \oint_{\partial\Omega} \mathbf{r} \times \mathbf{t} \mathrm{d}s = \int_{\Omega} [\mathbf{r} \times (\nabla \cdot \boldsymbol{\sigma} + \mathbf{b}) + \mathcal{E} : \boldsymbol{\sigma}^{\mathrm{T}}] \mathrm{d}v, \tag{1.74}$$

where $\mathcal{E}$ is the third-order permutation tensor.

Then, the law of conservation of angular momentum can be written as

$$\int_{\Omega} \left[\mathbf{r} \times \left(\nabla \cdot \boldsymbol{\sigma} + \mathbf{b} - \rho \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t}\right) + \mathcal{E} : \boldsymbol{\sigma}^{\mathrm{T}}\right] \mathrm{d}v = 0. \tag{1.75}$$

This equation holds for regions of arbitrary size. Combining it with the balance of momentum, we obtain the following:

$$\mathcal{E} : \boldsymbol{\sigma}^{\mathrm{T}} = 0, \tag{1.76}$$

$$\sigma_{ij} = \sigma_{ji}, \quad \boldsymbol{\sigma} = \boldsymbol{\sigma}^{\mathrm{T}}. \tag{1.77}$$

Thus, the balance of the angular momentum requires that the Cauchy stress tensor is a symmetric second-order tensor.

1.4.6 Balance of Mechanical Energy

The total energy of an object consists of kinetic energy and internal energy, while we consider only mechanical energy and neglect other forms of energy such as thermal, electrical, or chemical energy. The balance of mechanical energy can be stated as follows: the material time derivative of the total energy equals to the power of the external forces acting on the volume domain. This can be expressed as

$$\dot{K} + P_{\mathrm{int}} = P_{\mathrm{ext}}, \tag{1.78}$$

where K is the kinetic energy, P_{int} is the rate of internal mechanical work, and P_{ext} is the rate of external mechanical work.

The kinetic energy can be represented as

$$K = \int_{\Omega} \frac{1}{2} \rho \mathbf{v} \cdot \mathbf{v} \mathrm{d}v. \tag{1.79}$$

Using Reynolds Transport Theorem, the rate of the kinetic energy can be written as

$$\dot{K} = \frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega} \frac{1}{2} \rho \mathbf{v} \cdot \mathbf{v} \mathrm{d}v = \int_{\Omega} \rho \mathbf{v} \cdot \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} \mathrm{d}v. \tag{1.80}$$

The external mechanical work can be written as

$$\begin{aligned} P_{\mathrm{ext}} &= \int_{\Omega} \mathbf{b} \cdot \mathbf{v} \mathrm{d}v + \oint_{\partial\Omega} \mathbf{t} \cdot \mathbf{v} \mathrm{d}s \\ &= \int_{\Omega} \mathbf{b} \cdot \mathbf{v} \mathrm{d}v + \oint_{\partial\Omega} \boldsymbol{\sigma}\mathbf{n} \cdot \mathbf{v} \mathrm{d}s \\ &= \int_{\Omega} \mathbf{b} \cdot \mathbf{v} \mathrm{d}v + \int_{\Omega} \mathrm{div}(\boldsymbol{\sigma}^{\mathrm{T}}\mathbf{v}) \mathrm{d}v \\ &= \int_{\Omega} \mathbf{b} \cdot \mathbf{v} \mathrm{d}v + \int_{\Omega} (\mathrm{div}\boldsymbol{\sigma} \cdot \mathbf{v} + \boldsymbol{\sigma} : \mathrm{grad}\mathbf{v}) \mathrm{d}v. \end{aligned} \tag{1.81}$$

Combining the preceding four equations together with the balance of momentum (Eq. 1.69), it can be obtained that

$$P_{\text{int}} = \int_{\Omega} \boldsymbol{\sigma} : \text{grad}\mathbf{v}\text{d}v = \int_{\Omega} \boldsymbol{\sigma} : \mathbf{L}\text{d}v = \int_{\Omega} \boldsymbol{\sigma} : \mathbf{d}\text{d}v, \tag{1.82}$$

where $\mathbf{d}$ is the symmetric part of the spatial velocity gradient tensor.

Note. Various stress and strain tensors have been defined in the previous part. Stress and strain tensors defined within the same description method are thermodynamic conjugate variables. The rate of change in the deformation energy density can also be expressed using other stress and strain pairs using the relation: $J\boldsymbol{\sigma} : \mathbf{d} = \mathbf{P} : \dot{\mathbf{F}} = \mathbf{S} : \dot{\mathbf{E}}$.

1.4.7 Balance of Energy

For a system with both mechanical and thermal energy, the first law of thermodynamics should be stated as the increase in the kinetic energy and internal energy should be equal to the external mechanical power and thermal power, represented as

$$\frac{\text{d}}{\text{d}t}K + \frac{\text{d}}{\text{d}t}\int_{\Omega} e_{\text{c}}\text{d}V = P_{\text{ext}} + Q(t), \tag{1.83}$$

where e_{c} is the internal energy and $Q(t)$ is the rate of thermal work in the current configuration.

The kinetic energy and external mechanical power are the same as Eqs. (1.79) and (1.81). The thermal power can be defined as

$$\begin{aligned} Q(t) &= -\oint_{\partial\Omega} \mathbf{q} \cdot \mathbf{n}\text{d}s + \int_{\Omega} r\text{d}v \\ &= -\int_{\Omega} \text{div}\mathbf{q}\text{d}v + \int_{\Omega} r\text{d}v, \end{aligned} \tag{1.84}$$

where $\mathbf{q}$ is the heat flux through the surface, namely, heat per unit time and per unit area, and r is the heat source per unit time and per unit volume.

Using the same procedure, we can obtain the following form of the first law in current configuration as

$$\frac{\text{d}}{\text{d}t}\int_{\Omega} e_{\text{c}}\text{d}v = \int_{\Omega} (\boldsymbol{\sigma} : \mathbf{d} - \text{div}\mathbf{q} + r)\text{d}v. \tag{1.85}$$

The first law of thermodynamics can also be rewritten in the material configuration

$$\frac{\text{d}}{\text{d}t}\int_{\Omega_0} \left(\frac{1}{2}\rho_0 \mathbf{V} \cdot \mathbf{V} + e\right)\text{d}V = \int_{\Omega_0} (\mathbf{B} \cdot \mathbf{V} + R)\text{d}V + \oint_{\partial\Omega_0} (\mathbf{T} \cdot \mathbf{V} - \mathbf{Q} \cdot \mathbf{N})\text{d}S, \tag{1.86}$$

where e is the internal energy in reference configuration, R is the heat source in reference configuration, and $\mathbf{Q}$ is the heat flux in reference configuration.

Using the same procedure, the previous equation can be further reduced to the following form:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_0} e\mathrm{d}V = \int_{\Omega_0} (\mathbf{P} : \dot{\mathbf{F}} - \mathrm{Div}\mathbf{Q} + R)\mathrm{d}V. \tag{1.87}$$

Different from the first law in current configuration (Eq. 1.85), Eq. (1.87) can be written in the local form as

$$\dot{e} = \mathbf{P} : \dot{\mathbf{F}} - \mathrm{Div}\mathbf{Q} + R. \tag{1.88}$$

1.4.8 Entropy Inequality

We will then introduce the second law of thermodynamics, which governs the direction of energy transfer. The second law requires that in any process, the system evolves in a direction towards increasing entropy η. We first define the entropy flux $\mathbf{h}$ and the entropy source $\tilde{r}$ in the current configuration as

$$\mathbf{h} = \frac{\mathbf{q}}{T}, \quad \tilde{r} = \frac{r}{T}, \tag{1.89}$$

where T is the temperature.

The total entropy production of the whole system should not be negative, which gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega} \eta_c \mathrm{d}v + \oint_{\partial\Omega} \frac{\mathbf{q}}{T} \cdot \mathbf{n}\mathrm{d}s - \int_{\Omega} \frac{r}{T}\mathrm{d}v \geq 0, \tag{1.90}$$

where η_c is the entropy in the current configuration. This is the entropy inequality in the current configuration. The previous form cannot be directly written in the local form.

In order to obtain the local form of the entropy inequality, we need to write the entropy inequality in the reference configuration. Similarly, the entropy flux $\mathbf{H}$ and the entropy source $\widetilde{R}$ in the reference configuration can be represented as

$$\mathbf{H} = \frac{\mathbf{Q}}{T}, \quad \widetilde{R} = \frac{R}{T}. \tag{1.91}$$

The second law is then written as

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_0} \eta \mathrm{d}V + \oint_{\partial\Omega_0} \frac{\mathbf{Q}}{T} \cdot \mathbf{N}\mathrm{d}S - \int_{\Omega_0} \frac{R}{T}\mathrm{d}V \geq 0, \tag{1.92}$$

where η is entropy in the reference configuration.

Using the integral theorem, the equation can be rewritten as

$$\int_{\Omega_0} \left(\dot{\eta} + \frac{\mathrm{Div}\mathbf{Q}}{T} - \frac{\mathbf{Q} \cdot \mathrm{Grad}T}{T^2} - \frac{R}{T}\right)\mathrm{d}V \geq 0. \tag{1.93}$$

The local form of the second law of thermodynamics can then be written as

$$\dot{\eta} + \frac{\mathrm{Div}\mathbf{Q}}{T} - \frac{\mathbf{Q} \cdot \mathrm{Grad}T}{T^2} - \frac{R}{T} \geq 0. \tag{1.94}$$

Through elimination of the heat source R using Eq. (1.88), the preceding equation can be represented as

$$\mathbf{P} : \dot{\mathbf{F}} - \dot{e} + T\dot{\eta} - \frac{1}{T}\mathbf{Q} \cdot \mathrm{Grad}T \geq 0. \tag{1.95}$$

The last term in this equation is related to heat flow. Heat flows from the warmer to the colder region of a body. Hence, entropy production by conduction of heat must be nonnegative, i.e.

$$-\frac{1}{T}\mathbf{Q} \cdot \mathrm{Grad}T \geq 0. \tag{1.96}$$

Note. The Fourier's law is widely adopted for heat conduction. In the current configuration, the Fourier law for an isotropic body can be written as

$$\mathbf{q} = -k\mathrm{grad}T, \tag{1.97}$$

where $k > 0$ is the coefficient of thermal conductivity. The heat fluxes should be equal regardless of the reference or current configurations, which gives

$$\oint_{\partial\Omega_0} \mathbf{Q} \cdot \mathbf{N}\mathrm{d}S = \oint_{\partial\Omega} \mathbf{q} \cdot \mathbf{n}\mathrm{d}s. \tag{1.98}$$

Using Nanson's formula (Eq. 1.42), we obtain

$$\mathbf{Q} = J\mathbf{F}^{-1}\mathbf{q}. \tag{1.99}$$

We also have

$$\mathrm{grad}T = \mathrm{Grad}T\,\mathbf{F}^{-1}. \tag{1.100}$$

Combining the previous equations, we can obtain

$$\mathbf{Q} = -kJ\mathbf{F}^{-1}\mathrm{Grad}T\,\mathbf{F}^{-1}. \tag{1.101}$$

Substituting Eq. (1.101) into Eq. (1.96) leads to

$$\frac{kJ}{T}\mathbf{F}^{-1}\mathrm{Grad}T \cdot \mathbf{F}^{-1}\mathrm{Grad}T \geq 0. \tag{1.102}$$

Apparently, this holds true for all conditions with a nonnegative coefficient of thermal conductivity.

We further require that the remaining terms in Eq. (1.95) remain nonnegative, which gives

$$\mathbf{P} : \dot{\mathbf{F}} - \dot{e} + T\dot{\eta} \geq 0. \tag{1.103}$$

This equation is known as the Clausius–Planck inequality.

In continuum mechanics, the Helmholtz free energy density Ψ is also commonly used, which is a Legendre transformation of the internal energy as

$$\Psi = e - T\eta. \tag{1.104}$$

The Clausius–Planck inequality can be rewritten using the Helmholtz free energy density

$$\mathbf{P} : \dot{\mathbf{F}} - \dot{\Psi} - \eta \dot{T} \geq 0. \tag{1.105}$$

Neglecting the thermal effect, we can further obtain

$$\mathbf{P} : \dot{\mathbf{F}} - \dot{\Psi} \geq 0. \tag{1.106}$$

If the process is reversible, no dissipation occurs for the process. It can then be obtained that

$$\mathbf{P} : \dot{\mathbf{F}} - \dot{\Psi} = 0. \tag{1.107}$$

This will be the subject of hyperelasticity, which will be discussed further in Chapter 2.

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