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Overview of Continuum Mechanics

1.1 Definition of Tensor

1.1.1 Vectors and Tensors

In physics and mechanics, certain physical quantities are inherently independent of the choice of coordinate systems. A typical example is the vector, which represents a geometric quantity possessing both magnitude and direction. Common vector quantities in mechanics include displacement, velocity, force, and momentum, which are conventionally represented as directed line segments.

Vectors play a fundamental role across various natural sciences, including mathematics, physics, and engineering. Geometrically, a vector is often visualized as an arrowed line segment, where the length denotes its magnitude and the arrow indicates its direction. In contrast to vectors, scalar quantities—such as mass, temperature, and energy—possess magnitude but lack direction.

In mathematics, vectors are formally defined as elements of vector spaces (or linear spaces), which are abstract structures characterized by operations of addition and scalar multiplication. When interpreted physically, defining a vector requires the introduction of a norm and an inner product within the framework of Euclidean space, enabling the quantification of length and angle. For instance, the gradient of a scalar field is a vector field, while the derivative of a scalar with respect to another scalar remains a scalar.

Although a vector is an invariant geometric entity, its numerical components vary with the choice of coordinate system. To represent a vector, one must specify its components relative to a particular basis. These components transform according to specific rules–coordinate transformation laws—that preserve the underlying physical meaning of the vector. This transformation behavior underpins the vector's coordinate independence and leads naturally to the generalized concept of tensors.

Tensors generalize vectors and scalars, providing a powerful mathematical framework for formulating physical laws in a coordinate-independent manner. In many physical theories, it is necessary to express laws in terms of component quantities; however, the component values typically change under a transformation of the coordinate system. To ensure the validity of physical laws in any frame of reference, the transformation rules must apply consistently to all terms in an equation [1–3]. This principle guarantees the covariance of physical laws.

Constitutive Models of Solid Materials: From Mechanical Principles to Engineering Applications, First Edition.

From a physical standpoint, the components of a tensor correspond to measurable quantities in a given frame of reference, whereas the tensor itself represents the intrinsic physical quantity. Observations may differ between reference frames, but the form of the physical laws and equations remains invariant. To fully describe a tensorial physical quantity, one must understand not only its components in a particular frame but also how those components transform under changes of reference. It is this transformation behavior that preserves the form of physical equations and ensures their universal applicability.

1.1.2 Definition of Tensors

1. Definitions
   The introduced coordinate system represents the physical quantity $\boldsymbol{T}$ with some directional combination. Let the physical quantity $\boldsymbol{T}$ be represented in different Cartesian coordinate systems as:

$$\begin{aligned} \boldsymbol{T} &= T_{ij\cdots k}\boldsymbol{e}_i\boldsymbol{e}_j\cdots\boldsymbol{e}_k \\ \boldsymbol{T}' &= T'_{rs\cdots t}\boldsymbol{e}'_r\boldsymbol{e}'_s\cdots\boldsymbol{e}'_t \end{aligned} \tag{1.1}$$

where $\boldsymbol{e_i}\ \boldsymbol{e_j}...\boldsymbol{e_k}$ is the combined base vector required to represent the directionality of the physical quantity, and let the number of combined base vectors be n (i.e. n base vectors $\boldsymbol{e}$ multiplied). If its coordinate components meet the following coordinate conversion rules:

$$T_{ij\cdots k} = \beta_{ir'}\beta_{js'}\cdots\beta_{kt'}T'_{rs\cdots t} \tag{1.2}$$

Combined Eq (1.1) has

$$T'_{rs\cdots t}\boldsymbol{e}'_r\boldsymbol{e}'_s\cdots\boldsymbol{e}'_t = \beta_{ir'}\beta_{js'}\cdots\beta_{kt'}T'_{rs\cdots t}\boldsymbol{e}_i\boldsymbol{e}_j\cdots\boldsymbol{e}_k = T_{ij\cdots k}\boldsymbol{e}_i\boldsymbol{e}_j\cdots\boldsymbol{e}_k \tag{1.3}$$

i.e. $\boldsymbol{T} = \boldsymbol{T}'$ such a physical quantity is called a tensor, and the number n of the combined basis vectors is called the order of this tensor. It can be seen that tensors represent physical quantities with a certain combination of directions, which are independent of the choice of coordinate system [4–7].

For example, physical quantities such as temperature or mass density of an object are functions of position, independent of direction, and are called zero-order tensors, which are expressed as scalars. Other physical quantities are expressed by vectors, such as the displacement, velocity, acceleration of a particle, and the force exerted on an object, which can be expressed as $\boldsymbol{T} = T_i\boldsymbol{e_i}$, the number of combined base vectors $n = 1$, i.e. the vector is a first-order tensor. When the number of combined basis vectors is $n = 2$, it is called a second-order tensor, such as the inertia tensor representing the mass distribution of the object, the strain tensor representing the deformation at a point, and the corresponding stress tensor. Second-order tensors can be expressed as $\boldsymbol{T} = T_{ij}\boldsymbol{e_i}\boldsymbol{e_j}$. A tensor can be represented in a variety of ways, such as in $\boldsymbol{T}$ or other capitalized italic, bold. In the case of a given coordinate system, it can also be expressed in the form of omitting the base vector, as in the case of a second-order tensor T_{ij}, which represents a set of tensor components, where i and j are free indicators. The advantage of this representation is that it is expressed in scalar form, and when performing operations, it conforms to the well-known scalar algorithm [1, 5, 7]. Sometimes, for clarity, the second-order tensor $\boldsymbol{T}$ can also be represented by a matrix of 3×3:

$$\begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix} \tag{1.4}$$

Something like T_{12} represents a component of the tensor. Vectors can be represented in Ti or in arrays of 1×3:

$$\begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix} = \begin{bmatrix} T_1 & T_2 & T_3 \end{bmatrix}^T \tag{1.5}$$

2. Objectivity of Tensors
A tensor represents a physical quantity that is independent of the observer or the choice of reference frame. This invariance ensures that physical laws expressed in tensorial form remain valid in all coordinate systems, thereby satisfying the principle of objectivity required in physical formulations. Although the tensor itself is invariant, its component representation depends on the choice of basis vectors associated with a specific frame of reference. When the coordinate system changes, the basis vectors transform accordingly, and the tensor components adjust in such a way that the overall tensor remains unchanged. In this sense, a tensor is a covariant combination of basis and components. Thus, while the same tensor may have different component forms in different frames of reference, the underlying physical quantity it represents is preserved across all observers [1, 5, 7].

Therefore, it can be understood that if the tensor satisfies the corresponding coordinate transformation law, the tensor can be objective. For example, the scalar a, the first-order tensor (vector) $\boldsymbol{b}$, the second-order tensor $\boldsymbol{T}$ expressed in the reference line $Ox_1x_2x_3$ expressed under the reference line $Ox'_1x'_2\,x'_3$ need to satisfy the following relationship:

$$\boldsymbol{T}' = \boldsymbol{QTQ}^{\boldsymbol{T}} \tag{1.6}$$

So, at the level of coordinate transformation, the objectivity of the tensor and the indistinguishable property of the reference frame of the tensor are equivalent.

However, it cannot be understood that if the tensor satisfies the coordinate transformation rules of the tensor, it is said that the tensor has objectivity, and there will be some special tensors. For example, the deformation gradient tensor $\boldsymbol{F}$ is a second-order tensor categorically speaking, but its coordinate transformation rule is not $\boldsymbol{F}' = \boldsymbol{QFQ}^{\boldsymbol{T}}$, but $\boldsymbol{F}' = \boldsymbol{QF}$, this is because $\boldsymbol{F}$ is a special second-order tensor, also called a two-point tensor, because the base $\boldsymbol{e_i} \otimes \boldsymbol{E_j}$, the base $\boldsymbol{e_i}$ under the current configuration changes with time and is affected by the rotation tensor $\boldsymbol{Q}$; At the same time, the substrate $\boldsymbol{E_j}$ under the initial configuration is always unchanged, so it is not affected by $\boldsymbol{Q}$. Therefore, the two-point tensor $\boldsymbol{F}$ needs to satisfy the principle of objectivity, and the coordinate transformation law is the same as the first-order tensor, which is $\boldsymbol{F}' = \boldsymbol{QF}$.

At the same time, some tensors have objectivity, but their material time differentiation is not. For example, the Cauchy stress tensor σ has objectivity $\left(\boldsymbol{\sigma} = \boldsymbol{Q\sigma Q^T}\right)$, but the Cauchy stress rate tensor σ is not:

$$\boldsymbol{\sigma}' = \left(\mathbf{Q}\boldsymbol{\sigma}\mathbf{Q}^{\mathbf{T}}\right)' = \mathbf{Q}\boldsymbol{\sigma}\mathbf{Q}^{\mathbf{T}} + \mathbf{Q}\boldsymbol{\sigma}\mathbf{Q}^{\mathbf{T}} + \mathbf{Q}\boldsymbol{\sigma}\mathbf{Q}^{\mathbf{T}} \neq \mathbf{Q}\boldsymbol{\sigma}\mathbf{Q}^{\mathbf{T}} \tag{1.7}$$

Therefore, to ensure the objectivity of the Cauchy stress rate, several scholars have proposed objective stress rate formulations in tensorial form that are consistent with the transformation rules required for second-order tensors under a change of reference frame [8–10]. When establishing constitutive models—particularly those in rate form, such as hypoelastic models—it is essential to determine whether the employed tensor quantities are objective. If the material time derivative of a tensor is not objective, then any constitutive relation formulated using it may fail to be frame-indifferent, thereby violating a fundamental physical requirement.

Objective constitutive equations are typically expressed in rate form to capture time-dependent behavior, and ensuring the objectivity of stress and strain rates is crucial for the correctness of these models. If non-objective tensors are used directly in the formulation, it becomes challenging to construct frame-indifferent constitutive laws without introducing additional correction terms or structural complexity.

That said, it is not always necessary to use objective tensors in constitutive modeling. In certain cases, non-objective tensors can still yield physically meaningful results if the constitutive equations are constructed in terms of scalar invariants (e.g. norm or trace) or functions of tensors that remain invariant under observer transformations. In practice, many constitutive models adopt such an approach, leveraging observer-invariant quantities rather than explicitly ensuring the objectivity of every tensor involved.

For example, Green's elastic constructs use a tensor of the observer's invariant quantity, which is like:

$$S = \rho^{\,0}\frac{\partial \psi^{\mathrm{e}}}{\partial E} = 2\rho^{0}\frac{\partial \psi^{\mathrm{e}}}{\partial C} \tag{1.8}$$

In the Eq (1.8), $S,\ E\ and\ C$ are all observer invariants, and can also be written as $S^{*} = \bar{\rho}\ \frac{\partial \psi^{e}\left(E^{*}\right)}{\partial E}$, for $S = S'$, this is a constitutive equation that satisfies the objectivity of physical laws.

1.2 Coordinate Transformations

1.2.1 Summation Convention

Consider the following sums:

$$s = a_1 x_1 + a_2 x_2 + \ldots + a_n x_n. \tag{1.9}$$

Equation (1.9) can be simplified using a summation notation:

$$s = \sum_{i=1}^{n} a_i x_i \tag{1.10}$$

At the same time, it is possible to use any symbol instead of i as a free indicator, such as

$$s = \sum_{j=1}^{n} a_j x_j, \quad s = \sum_{m=1}^{n} a_m x_m, \quad s = \sum_{k=1}^{n} a_k x_k \tag{1.11}$$

Whether it is i in the equation, or j, m, k, it is a dummy metric because the sum is independent of the letters used for indexing. The equation can be further simplified if the following convention is adopted: when an index is repeated once, it is a dummy metric that represents the sum of the index with the integers $1,\ 2, \ldots, n$. This convention is called Einstein's summation convention.

Using this convention, the equation can be simply written:

$$s = a_i x_i \text{ or } s = a_j x_j \text{ or } s = a_m x_m \ldots \tag{1.12}$$

It is important to emphasize that expressions such as $a_i b_i x_i$ or $a_m b_m x_m$ are not defined in this convention. This means that when summation conventions are used, indicator symbols should not be repeated more than once. Therefore, the expression of the summation form:

$$\sum_{i=1}^{n} a_i b_i x_i \tag{1.13}$$

must retain its sum symbol.

For ease of understanding, n in the sum formula is taken as 3 in the following text:

$$a_i x_i = a_1 x_1 + a_2 x_2 + a_3 x_3, \quad a_{ii} = a_{11} + a_{22} + a_{33}. \tag{1.14}$$

The summation convention can obviously be used to denote the sum of two or the sum of three things, etc. For example:

$$\alpha = \sum_{i=1}^{3} \sum_{j=1}^{3} a_{ij} x_i x_j \tag{1.15}$$

or

$$\alpha = a_{ij} x_i x_j. \tag{1.16}$$

Equation (1.16) can be fully expanded as:

$$\begin{aligned} \alpha = a_{ij} x_i x_j = {} & a_{11} x_1 x_1 + a_{12} x_1 x_2 + a_{13} x_1 x_3 + a_{21} x_2 x_1 + a_{22} x_2 x_2 \\ & + a_{23} x_2 x_3 + a_{31} x_3 x_1 + a_{32} x_3 x_2 + a_{33} x_3 x_3 \end{aligned} \tag{1.17}$$

Similarly, the tag symbol $a_{ijk} x_i x_j x_k$ represents the triple sum of 27 items, i.e.

$$\sum_{i=1}^{3} \sum_{j=1}^{3} \sum_{k=1}^{3} a_{ijk} x_i x_j x_k = a_{ijk} x_i x_j x_k. \tag{1.18}$$

Consider the following set of equations:

$$\begin{aligned} x_1' &= a_{11}x_1 + a_{12}x_2 + a_{13}x_3, \\ x_2' &= a_{21}x_1 + a_{22}x_2 + a_{23}x_3, \\ x_3' &= a_{31}x_1 + a_{32}x_2 + a_{33}x_3, \end{aligned} \tag{1.19}$$

The use of summation conventions can be abbreviated as:

$$\begin{aligned} x_1' &= a_{1m}x_m, \\ x_2' &= a_{2m}x_m, \\ x_3' &= a_{3m}x_m, \end{aligned} \tag{1.20}$$

Further simplification can be obtained:

$$x_i' = a_{im}x_m, \quad i = 1, 2, 3 \tag{1.21}$$

An indicator that appears only once in each term of the equation, such as indicator i in Eq. (1.21), is called a free exponent. In general, a free exponent is an integer of 1, 2, or 3. Thus, Eq. (1.21) is an abbreviation of three equations, each with the sum of the three to the right.

1.2.2 Kronecker Increment

The Kronecker delta, denoted by $\boldsymbol{\delta_{ij}}$, can be defined as:

$$\delta_{ij} = \begin{cases} 1 & \text{if} : i = j \\ 0 & \text{if} : i \neq j \end{cases} \tag{1.22}$$

i.e.

$$\delta_{11} = \delta_{22} = \delta_{33} = 1, \delta_{12} = \delta_{13} = \delta_{21} = \delta_{23} = \delta_{31} = \delta_{32} = 0 \tag{1.23}$$

In other words, Kronecker's delta matrix is an identity matrix:

$$[\delta_{ij}] = \begin{bmatrix} \delta_{11} & \delta_{12} & \delta_{13} \\ \delta_{21} & \delta_{22} & \delta_{23} \\ \delta_{31} & \delta_{32} & \delta_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{1.24}$$

According to the characteristics of the indicator and the summation convention, the Kronecker increment has the following properties.

Nature 1:

$$\delta_{ii} = \delta_{11} + \delta_{22} + \delta_{33} = 1 + 1 + 1 \tag{1.25}$$

i.e.

$$\delta_{ii} = 3 \tag{1.26}$$

Nature 2:

$$\delta_{im}a_m = a_i \tag{1.27}$$

Nature 3:

$$\begin{aligned}\delta_{1m}T_{mj} &= \delta_{11}T_{1j} + \delta_{12}T_{2j} + \delta_{13}T_{3j} = T_{1j}\\ \delta_{2m}T_{mj} &= \delta_{21}T_{1j} + \delta_{22}T_{2j} + \delta_{23}T_{3j} = T_{2j}\\ \delta_{3m}T_{mj} &= \delta_{31}T_{1j} + \delta_{32}T_{2j} + \delta_{33}T_{3j} = T_{3j}\end{aligned} \tag{1.28}$$

i.e.

$$\delta_{im}T_{mj} = T_{ij} \tag{1.29}$$

Nature 4: if $\boldsymbol{e_1}$, $\boldsymbol{e_2}$, and $\boldsymbol{e_3}$ are unit vectors perpendicular to each other, then there is

$$\boldsymbol{e}_i \cdot \boldsymbol{e}_j = \delta_{ij} \tag{1.30}$$

1.2.3 Coordinate Transformations

In three-dimensional space, three lines x_1, x_2 and x_3 orthogonal to each other at the origin O form a Cartesian coordinate system. $\boldsymbol{e_1}$, $\boldsymbol{e_2}$, and $\boldsymbol{e_3}$ represent the direction of these three lines, respectively, and their length is 1, which is called the base vector. It is customary to use the right-hand coordinate system, as shown in Figure 1.1. The dot product of any two of its base vectors is

$$\boldsymbol{e_i} \cdot \boldsymbol{e}_j = \delta_{ij} = \delta_{ji} = \begin{cases} 1, i = j \\ 0, i \neq j \end{cases} \tag{1.31}$$

where $\boldsymbol{\delta_{ij}}$ is the Kronecker operator, and the subscripts i and j can be freely selected from 1, 2, and 3, which is called the free indicator. If expressed in matrix form, there is

$$[\delta_{ij}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{1.32}$$

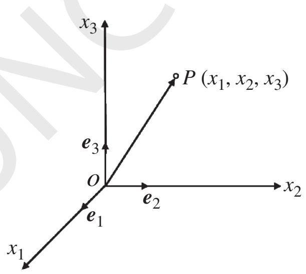

Figure 1.1 Cartesian coordinate system and basis vectors.

$$x = x_1\boldsymbol{e}_1 + x_2\boldsymbol{e}_2 + x_3\boldsymbol{e}_3 \tag{1.33}$$

According to Einstein's summation convention, Eq. (1.33) can be abbreviated as:

$$x = x_i\boldsymbol{e_i} \tag{1.34}$$

where the subscript i is repeated and is called a dumb mark. In a three-dimensional Euclidean space, the dumb mark i is summed by 1, 2, and 3, respectively, so that the dumb mark can be represented by any letter. For example

$$\delta_{ii} = \delta_{11} + \delta_{22} + \delta_{33} = 3 \tag{1.35}$$

Let another set of right-hand Cartesian coordinate systems $Ox_1{}'x_2{}'x_3{}'$ intersect at the origin O, and its base vectors are e_1', e_2', e_3'. The dot product of the two Cartesian coordinate systems base vectors e_i and e_i' is the direction cosine of the two rectilinear directions:

$$\beta_{ij'} = \boldsymbol{e_i} \cdot \boldsymbol{e_j'} = \cos\left(x_i, x_j'\right), \beta_{i'j} = \boldsymbol{e_i'} \cdot \boldsymbol{e_j} = \cos\left(x_i', x_j\right) \tag{1.36}$$

It is denoted by a matrix as

$$[\beta_{ij'}] = \begin{bmatrix} \cos\left(x_1, x_1'\right) & \cos\left(x_1, x_2'\right) & \cos\left(x_1, x_3'\right) \\ \cos\left(x_2, x_1'\right) & \cos\left(x_2, x_2'\right) & \cos\left(x_2, x_3'\right) \\ \cos\left(x_3, x_1'\right) & \cos\left(x_3, x_2'\right) & \cos\left(x_3, x_3'\right) \end{bmatrix} \tag{1.37}$$

where $[\boldsymbol{\beta}_{i'j}]$ is the transpose matrix of $[\boldsymbol{\beta}_{ij'}]$, i.e. $[\boldsymbol{\beta}_{i'j}] = [\boldsymbol{\beta}_{ij'}]^T$, or $\boldsymbol{\beta}_{i'j} = \boldsymbol{\beta}_{j'i}$. Since $[\boldsymbol{\beta}_{ij'}]$ $[\boldsymbol{\beta}_{j'k}] == [\boldsymbol{\delta_{ik}}]$, it is shown that $[\boldsymbol{\beta}_{ij'}]$ is an orthogonal matrix, i.e. $[\boldsymbol{\beta}_{i'j}] = [\boldsymbol{\beta}_{ij'}]^{-1}$.

Thus, the transformation relationship between the two sets of basis vectors satisfies the transformation:

$$\boldsymbol{e}_i = \beta_{ij'}\boldsymbol{e}_j' \text{或} \boldsymbol{e}_i' = \beta_{i'j}\boldsymbol{e}_j \tag{1.38}$$

The position of the P point can also be expressed by the vector diameter $\boldsymbol{x}'$, $\boldsymbol{x}' = \boldsymbol{x}$, by Eq. (1.38), there is

$$x_i\boldsymbol{e}_i = x_j'\boldsymbol{e}_j' = x_j'\beta_{j'i}\boldsymbol{e}_i = \beta_{ij'}x_j'\boldsymbol{e}_i \tag{1.39}$$

Comparison Eq. (1.45) is available on both sides

$$\begin{aligned} &x_i = \beta_{ij'}x_j', \text{ or} \\ &x_i' = \beta_{i'j}x_j \end{aligned} \tag{1.40}$$

Equation (1.46) is the coordinate transformation relationship between two sets of coordinate systems, $[\boldsymbol{\beta}_{i'j}]$ or $[\boldsymbol{\beta}_{ij'}]$ is called the coordinate transformation matrix.

1.2.4 Permutation Symbols

The permutation symbol represented by $\boldsymbol{\varepsilon}_{ijk}$ is defined as follows:

$$\varepsilon_{ijk} = \begin{Bmatrix} 1 \\ -1 \\ 0 \end{Bmatrix} \text{Traverse the order of } 1,\ 2,\ 3 \text{ according to } i,\ j,\ k \tag{1.41}$$

i.e.

$$\begin{aligned} &\varepsilon_{123} = \varepsilon_{231} = \varepsilon_{312} = 1 \\ &\varepsilon_{213} = \varepsilon_{321} = \varepsilon_{132} = -1 \\ &\varepsilon_{111} = \varepsilon_{112} = \varepsilon_{222} = \dots = 0 \end{aligned} \tag{1.42}$$

It can be recorded as:

$$\varepsilon_{ijk} = \varepsilon_{jki} = \varepsilon_{kij} = -\varepsilon_{jik} = -\varepsilon_{kji} - \varepsilon_{ikj} \tag{1.43}$$

From the vector mixture product, it can be seen that $(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c})$ denotes the volume of a parallelepiped composed of $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$. Thus, the displacement symbol can be represented by the mixed product of the Cartesian coordinate system base vector

$$\boldsymbol{e}_{\boldsymbol{ijk}} = \left(\boldsymbol{e}_{\boldsymbol{i}}, \boldsymbol{e}_{\boldsymbol{j}}, \boldsymbol{e}_{\boldsymbol{k}}\right) \tag{1.44}$$

If $\{\boldsymbol{e}_1,\ \boldsymbol{e}_2,\ \boldsymbol{e}_3\}$ is a set of vectors that follow the right-hand rule, then there is

$$\boldsymbol{e}_1 \times \boldsymbol{e}_2 = \boldsymbol{e}_3, \boldsymbol{e}_2 \times \boldsymbol{e}_1 = -\boldsymbol{e}_3, \boldsymbol{e}_2 \times \boldsymbol{e}_3 = \boldsymbol{e}_1, \boldsymbol{e}_3 \times \boldsymbol{e}_2 = -\boldsymbol{e}_1 \tag{1.45}$$

It can also be shortened as:

$$\boldsymbol{e}_{\boldsymbol{i}} \times \boldsymbol{e}_{\boldsymbol{j}} = \varepsilon_{ijk}\boldsymbol{e}_{\boldsymbol{k}} = \varepsilon_{jki}\boldsymbol{e}_{\boldsymbol{k}} = \varepsilon_{kij}\boldsymbol{e}_{\boldsymbol{k}} \tag{1.46}$$

Now there are two certain vectors

$$\boldsymbol{a} = a_i\boldsymbol{e}_{\boldsymbol{i}} \tag{1.47}$$

$$\boldsymbol{b} = b_i\boldsymbol{e}_{\boldsymbol{i}} \tag{1.48}$$

Since the results of the vector cross product are in different directions, they can be obtained

$$\boldsymbol{a} \times \boldsymbol{b} = (a_i\boldsymbol{e}_{\boldsymbol{i}}) \times \left(b_j\boldsymbol{e}_{\boldsymbol{j}}\right) = a_i b_j \left(\boldsymbol{e}_{\boldsymbol{i}} \times \boldsymbol{e}_{\boldsymbol{j}}\right) = a_i b_j \varepsilon_{ijk}\boldsymbol{e}_{\boldsymbol{k}} \tag{1.49}$$

1.3 Basic Operations of Tensors

1.3.1 Vector Operations

Vectors are physical quantities characterized by both magnitude and direction, and are commonly used to represent displacement, velocity, force, and other directional phenomena in physics. Geometrically, vectors are typically illustrated as directed line segments (arrows), where the length indicates magnitude and the orientation indicates direction.

Mathematically, vectors are elements of a vector space and can be manipulated through well-defined operations, including vector addition, scalar multiplication, dot product (also known as inner product), and cross product. These operations obey specific algebraic rules and are foundational to the formulation of physical laws and mathematical models in mechanics and engineering.

Vector Addition: Add the length and direction of the two vectors to get a new vector. For example:

$$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC} \tag{1.50}$$

where, $\overrightarrow{AB}$, $\overrightarrow{BC}$, $\overrightarrow{AC}$ are all vectors, representing the displacement from point A to point B, from point B to point C, and from point A to point C, respectively. Add $\overrightarrow{AB}$ and $\overrightarrow{BC}$ to give $\overrightarrow{AC}$. This process can be represented by the law of triangles or the rule of horizontal quadrilaterals.

Vector Subtraction: Reverse the length and direction of one vector to the length and direction of another, and then do the addition operation. For example:

$$\overrightarrow{AB} - \overrightarrow{AC} = \overrightarrow{CB} \tag{1.51}$$

Where $\overrightarrow{AB}$, $\overrightarrow{AC}$, $\overrightarrow{CB}$ represent the displacement from point A to point B, from point A to point C, and from point C to point B, respectively. Add $\overrightarrow{AB}$ and $\left(-\overrightarrow{AC}\right)$ to get $\overrightarrow{CB}$.

Quantity Product: Also known as dot product, it is used to calculate the angle between two vectors and the projection between them. For example:

$$\boldsymbol{a} \cdot \boldsymbol{b} = |\boldsymbol{a}|\,|\boldsymbol{b}| \cos\theta = a_x b_x + a_y b_y \tag{1.52}$$

where $\boldsymbol{a}$ and $\boldsymbol{b}$ are vectors, $|\boldsymbol{a}|$ and $|\boldsymbol{b}|$ are the lengths of $\boldsymbol{a}$ and $\boldsymbol{b}$, respectively. θ denotes the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$; The value of $\boldsymbol{a} \cdot \boldsymbol{b}$ is equal to the sum of the multiplication of the components of $\boldsymbol{a}$ and $\boldsymbol{b}$ in the same direction, that is, the projections of $\boldsymbol{a}$ and $\boldsymbol{b}$ are multiplied and then summed; a_i and b_i represent the projections of vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ on the i axis ($i = x, y, z$) in the spatial rectangular coordinate system, respectively.

Cross Product: Also known as a vector product, the result is a vector quantity. Used to calculate the vector between two vectors perpendicular to the plane in which they are located. For example:

$$\boldsymbol{a} \times \boldsymbol{b} = |\boldsymbol{a}|\,|\boldsymbol{b}| \sin\theta \vec{n} = \left(a_y b_z - a_z b_y, a_z b_x - a_x b_z, a_x b_y - a_y b_x\right) \tag{1.53}$$

where $\boldsymbol{a}$ and $\boldsymbol{b}$ are vectors, $|\boldsymbol{a}|$ and $|\boldsymbol{b}|$ are the lengths of $\boldsymbol{a}$ and $\boldsymbol{b}$, respectively. θ denotes the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$. $\vec{n}$ denotes the normal vector of the plane in which they are placed. The value of $\boldsymbol{a} \times \boldsymbol{b}$ is equal to the product of the area on the plane where $\boldsymbol{a}$ and $\boldsymbol{b}$ are located and the product of $\vec{n}$.

Mixed Product: The mixed product $(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c})$ of the vector $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ is a scalar quantity.

$$(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}) = \boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c}) = \begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix} \tag{1.54}$$

The vector positions of the mixed product $(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c})$ can be rotated:

$$\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c}) = \boldsymbol{b} \cdot (\boldsymbol{c} \times \boldsymbol{a}) = \boldsymbol{c} \cdot (\boldsymbol{a} \times \boldsymbol{b}) \tag{1.55}$$

The mixed product $(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c})$ represents the volume of a parallelepiped composed of $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$.

1.3.2 Operations of Tensors

1. **The Addition of Tensors**
   When two tensors of the same order are added, their corresponding components are added, and the basis vector remains unchanged, and the tensor of the same order is obtained:

$$\boldsymbol{A} + \boldsymbol{B} = \boldsymbol{T} \tag{1.56}$$

For second-order tensors and fourth-order tensors:

$$A_{ij} + B_{ij} = T_{ij}$$
$$A_{ijkl} + B_{ijkl} = T_{ijkl} \tag{1.57}$$

2. **The Juxtaposition of Tensors**
   Definition: The juxtaposition of two tensors $\boldsymbol{A}$ and $\boldsymbol{B}$ is the juxtaposition of the basis vectors of the two tensors, and their corresponding components are multiplied to obtain a higher-order tensor whose order is equal to the sum of the orders $\boldsymbol{A}$ and $\boldsymbol{B}$
   For example, the m-order tensor $\boldsymbol{A}$ is the juxtaposition of the nth-order tensor

$$\boldsymbol{AB} = \boldsymbol{T} \tag{1.58}$$

where $\boldsymbol{T}$ is a tensor of order m + n.
Nature 1: If $\boldsymbol{A}$ and $\boldsymbol{B}$ are second-order tensors, then $\boldsymbol{T}$ is a fourth-order tensor, i.e. there is

$$A_{ij}B_{kl}\boldsymbol{e_i e_j e_k e_l} = T_{ijkl}\boldsymbol{e_i e_j e_k e_l} \tag{1.59}$$
$$\Rightarrow A_{ij}B_{kl} = T_{ijkl}$$

Nature 2: The scalar k is multiplied by the second-order tensor $\boldsymbol{B}$, and the resulting tensor $\boldsymbol{T}$ is still a second-order tensor:

$$k\mathbf{B} = \boldsymbol{T} \tag{1.60}$$
$$\Rightarrow kB_{ij} = T_{ij}$$

3. **The Dot Product of Tensors**
   Definition: The dot product of two tensors is to multiply two tensors together, and then dot product the arbitrarily specified base vectors of the two tensors. If not specified in advance, the dot product of the two basis vectors closest to the two tensors. At the same time, the tensor basis vector after the dot product is reduced by two.
   For example, the dot product of a m-order tensor $\boldsymbol{A}$ and n-order tensor $\boldsymbol{B}$:

$$\boldsymbol{A} \times \boldsymbol{B} = \boldsymbol{T} \tag{1.61}$$

where $\boldsymbol{T}$ is a tensor of order $m + n - 2$. Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be second-order tensors, then $\boldsymbol{A} \cdot \boldsymbol{B} = \boldsymbol{T}$ is also a second-order tensor:

$$\boldsymbol{A} \cdot \boldsymbol{B} = \left(A_{ij}\boldsymbol{e_i e_j}\right) \cdot (B_{kl}\boldsymbol{e_k e_l}) = A_{ij}\delta_{jk}B_{kl}\boldsymbol{e_i e_l} = A_{ik}B_{kl}\boldsymbol{e_i e_l} = T_{il}\boldsymbol{e_i e_l} = \boldsymbol{T}$$

namely:

$$A_{ik}B_{kl} = T_{il} \tag{1.62}$$

The double dot product is the product of two tensors after multiplying them together, and then dotting their base vectors twice in order. For example, if $\boldsymbol{A}$ and $\boldsymbol{B}$ are second-order tensors, then $\boldsymbol{AB} = k$ is a zero-order tensor (i.e. a scalar):

$$\boldsymbol{A} : \boldsymbol{B} = \left(A_{ij}e_ie_j\right) : (B_{kl}e_ke_l) = A_{ij}\delta_{ik}\delta_{jl}B_{kl} = A_{kj}B_{kj} = k \tag{1.63}$$

The dot product of the vector $\boldsymbol{U}$ and $\boldsymbol{V}$ is

$$\boldsymbol{U}\cdot \boldsymbol{V} = (U_ie_i)\cdot\left(V_je_j\right) = U_iV_i = k \tag{1.64}$$

Geometric meaning of scalar k: $k = |\boldsymbol{U}||\boldsymbol{V}|\cos\theta$. where θ is the angle between the vector $\boldsymbol{U}$ and $\boldsymbol{V}$, $|\boldsymbol{U}|$and $|\boldsymbol{V}|$, respectively, are the absolute values of the two vectors. When $\boldsymbol{V}$ is the unit vector, k represents the projection of the vector $\boldsymbol{U}$ in the $\boldsymbol{V}$ direction; When $\boldsymbol{U}$ and $\boldsymbol{V}$ are nonzero vectors, and $\boldsymbol{U}\cdot\boldsymbol{V} = 0$, then $\boldsymbol{U}$ is orthogonal to $\boldsymbol{V}$ and vice versa.

4. **Tensor Condensation**
In the union vector notation of tensors, if a dot product is performed on one of the two basis vectors, the original tensor will be reduced by two orders, and this process is called tensor condensation, such as:

$$\boldsymbol{a}\cdot\boldsymbol{bcd} = (\boldsymbol{a}\cdot\boldsymbol{b})\boldsymbol{cd} \tag{1.65}$$

$$\boldsymbol{ab}\cdot\boldsymbol{cd} = \boldsymbol{a}(\boldsymbol{b}\cdot\boldsymbol{c})\boldsymbol{d} \tag{1.66}$$

$$\boldsymbol{abc}\cdot\boldsymbol{d} = \boldsymbol{ab}(\boldsymbol{c}\cdot\boldsymbol{d}) \tag{1.67}$$

and other forms. The dot product in the right parentheses of the above is a number, so they are reduced to second-order parallel vectors. The second-order union is condensed and merged into a number, and the number can be regarded as the zero-order union.

5. **The Quotient of Tensors**
If a physical quantity $\boldsymbol{T}$ is dotted with any tensor $\boldsymbol{B}$ to give a tensor $\boldsymbol{A}$, then the physical quantity $\boldsymbol{T}$ must be a tensor, and the tensor $\boldsymbol{T}$ is called the quotient of tensor $\boldsymbol{A}$ relative to tensor $\boldsymbol{B}$. For example, if the dot product of a physical quantity $\boldsymbol{T}$ and an arbitrary vector $\boldsymbol{V}$ gives the n-1 tensor $\boldsymbol{A}$, then the physical quantity $\boldsymbol{T}$ must be an n-order tensor.
In the same way, if the product of a physical quantity $\boldsymbol{T}$ and any second-order tensor $\boldsymbol{B}$ gives the n-second order tensor $\boldsymbol{A}$, then the physical quantity $\boldsymbol{T}$ must be an nth-order tensor

6. **Transpose of Tensors**
Definition: The transpose $\boldsymbol{T}^T$ of the second-order tensor $\boldsymbol{T} = T_{ij}\boldsymbol{e_i}\boldsymbol{e_j}$ is defined as:

$$\boldsymbol{T}^T = T_{ij}\boldsymbol{e_i}\boldsymbol{e_j} \tag{1.68}$$

Nature 1:

$$\left(\boldsymbol{T}^T\right)^T = \boldsymbol{T} \tag{1.69}$$

Nature 2: if $\boldsymbol{T}^T = \boldsymbol{T}$, or $T_{ij} = T_{ji}$, then $\boldsymbol{T}$ is said to be a symmetric tensor.

Nature 3: Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be second-order tensors, and $\boldsymbol{U}$ be vectors

$$(\boldsymbol{A} \cdot \boldsymbol{B})^T = \boldsymbol{B^T} \cdot \boldsymbol{A^T} \tag{1.70}$$

$$\boldsymbol{A} \cdot \boldsymbol{U} = \boldsymbol{U} \cdot \boldsymbol{A^T} \tag{1.71}$$

1.3.3 Isotropic Tensors

Definition: If each component of a quantity remains unchanged in different coordinate systems, the tensor is said to be isotropic tensor.

Nature 1: The scalar is a zero-order tensor, which has nothing to do with the choice of coordinates, so any scalar is an isotropic tensor. The product of any scalar k and the unit tensor δ' is an isotropic tensor.

The second-order tensor $\delta = \delta_{ij}\boldsymbol{e}_i\boldsymbol{e}_j$ with δ as the coordinate component is called the unit tensor. In different coordinate systems, $\delta' = \delta'_{\boldsymbol{ij}}\boldsymbol{e'}_{\boldsymbol{i}}\boldsymbol{e'}_{\boldsymbol{j}}$, and $\delta'_{ij} = \boldsymbol{e'}_{\boldsymbol{i}} \cdot \boldsymbol{e'}_{\boldsymbol{j}} = \delta_{ij}$. Therefore, the product of the arbitrary scalar k and the unit tensor δ is an isotropic tensor.

Nature 2: Vectors are first-order tensors with no isotropic tensors.

Nature 3: A second-order isotropic tensor must be expressed as a form of $k\delta$ or $k\delta_{ij}$.

Proof that if T is a second-order isotropic tensor, then there is

$$T_{ij} = \beta_{ir'}\beta_{js'}T'_{rs} = T'_{ij} \tag{1.72}$$

Rotate the coordinate system 90° around the x_3 axis, and the coordinate conversion matrix is

$$[\beta_{ij'}] = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{1.73}$$

Substitution (1.72), there is $T_{11} = T'_{22} = T'_{11}, T_{22} = T'_{11} = T'_{22}$, so $T_{11} = T_{22}$. Rotate the coordinate system 90°around the x_2 axis to give $T_{11} = T_{33}$, so that $T_{11} = T_{22} = T_{33} = k$; In the same way, if the coordinate system is rotated 180° about the x_3 axis, there is $T_{13} = -T'_{13} = T'_{13}$, so $T_{13} = 0$, and similarly, when $i \neq j$, $T_{ij} = 0$, i.e. $T_{ij} = k\delta_{ij}$.

Definition: The third-order tensor $\varepsilon = e_{ijl}\boldsymbol{e_i}\ \boldsymbol{e_j}\boldsymbol{e_l}$ with the permutation symbol e_{ijl} as the coordinate component is called the transformation tensor.

Nature 4: A third-order isotropic tensor must be expressed as something like $k\varepsilon$ or ke_{ijl}, where k is an arbitrary scalar.

There are three basic forms of fourth-order isotropic tensors

$$\boldsymbol{I}^{(0)} = \delta_{ij}\delta_{kl}\boldsymbol{e_i}\boldsymbol{e_j}\boldsymbol{e_k}\boldsymbol{e_l}, \boldsymbol{I}^{(1)} = \delta_{ik}\delta_{jl}\boldsymbol{e_i}\boldsymbol{e_j}\boldsymbol{e_k}\boldsymbol{e_l}, \boldsymbol{I}^{(2)} = \delta_{il}\delta_{jk}\boldsymbol{e_i}\boldsymbol{e_j}\boldsymbol{e_k}\boldsymbol{e_l} \tag{1.74}$$

The fourth-order isotropic tensor $\boldsymbol{I}^{(i)}$ must be expressed in the form of $a\boldsymbol{I}^{(0)} + b\boldsymbol{I}^{(1)} + c\boldsymbol{I}^{(2)}$, or

$$I^{(i)}_{ijkl} = a\delta_{ij}\delta_{kl} + b\delta_{ik}\delta_{jl} + c\delta_{il}\delta_{jk} \tag{1.75}$$

where a, b, and c are arbitrary scalars.

1.3.4 Tensor Functions and Calculus Operations

1. **A Function of Tensors**
In algebra, matrices can serve as elements of certain functions. For example, the k-power of the nth order square, i.e. the matrix itself multiplied by $k-1$ times, is still an nth order square:

$$[\boldsymbol{T}]^2 = [\boldsymbol{T}]\,[\boldsymbol{T}] \tag{1.76}$$

$$[\boldsymbol{T}]^{\boldsymbol{k}} = \underbrace{[\boldsymbol{T}]\,[\boldsymbol{T}]\cdots[\boldsymbol{T}]}_{\boldsymbol{k}\times[\boldsymbol{T}]} \tag{1.77}$$

Constructing the kth polynomial of the matrix $\boldsymbol{T}$, which is still an n-order square matrix, denoted $\boldsymbol{H}$, defines the matrix $\boldsymbol{H}$ as a function of the matrix $\boldsymbol{T}$:

$$[\boldsymbol{H}] = \boldsymbol{f}([\boldsymbol{T}]) = \boldsymbol{c}_0\,[1] + \boldsymbol{c}_1\,[\boldsymbol{T}] + \boldsymbol{c}_2\,[\boldsymbol{T}]^2 + \cdots + \boldsymbol{c}_{\boldsymbol{k}}\,[\boldsymbol{T}]^{\boldsymbol{k}} \tag{1.78}$$

It is also defined that the second-order tensor $\boldsymbol{H}$ is a function of the second-order tensor $\boldsymbol{T}$ as an independent variable:

$$\boldsymbol{H} = \boldsymbol{f}(\boldsymbol{T}) = \boldsymbol{c}_0\boldsymbol{G} + \boldsymbol{c}_1\boldsymbol{T} + \boldsymbol{c}_2\boldsymbol{T}^2 + \cdots + \boldsymbol{c}_{\boldsymbol{k}}\boldsymbol{T}^{\boldsymbol{k}} \tag{1.79}$$

Equation (1.79) actually reflects the functional relationship between the elements of the matrix $\boldsymbol{H}$ and the matrix $\boldsymbol{T}$, and the component of this equation:

$$\boldsymbol{H}_{\boldsymbol{ij}} = \boldsymbol{c}_0\boldsymbol{\delta}_{\boldsymbol{ij}} + \boldsymbol{c}_1\boldsymbol{T}_{\boldsymbol{ij}} + \boldsymbol{c}_2\boldsymbol{T}_{\boldsymbol{il}}\boldsymbol{T}_{\boldsymbol{li}} + \cdots + \boldsymbol{c}_k\boldsymbol{T}_{\boldsymbol{il}_1}\boldsymbol{T}_{\boldsymbol{l}_1\boldsymbol{l}_2}\cdots\boldsymbol{T}_{\boldsymbol{l}_{\boldsymbol{k}-1}\boldsymbol{j}} \tag{1.80}$$

The functions listed above are polynomials. In general, if a tensor $\boldsymbol{H}$ (scalar, vector, tensor) depends on n tensors $\boldsymbol{T}_1, \boldsymbol{T}_2, \ldots, \boldsymbol{T}_n$ (vector, tensor), that is, when $\boldsymbol{T}_1, \boldsymbol{T}_2, \ldots, \boldsymbol{T}_n$ are given, $\boldsymbol{H}$ can be determined correspondingly, then $\boldsymbol{H}$ is said to be the tensor function of tensors $\boldsymbol{T}_1, \boldsymbol{T}_2, \ldots, \boldsymbol{T}_n$, denoted as $\boldsymbol{H}$ is the tensor function of tensors $\boldsymbol{T}_1, \boldsymbol{T}_2, \ldots, \boldsymbol{T}_n$.

2. **Differentiation and Derivatives of Tensors**
The following research methods are applicable to tensor field functions of any order (n order) in three-dimensional space, namely:

$$\boldsymbol{T} = \boldsymbol{T}(\boldsymbol{r}) \tag{1.81}$$

Its parallel vector expression is

$$\boldsymbol{T}(\boldsymbol{r}) = \boldsymbol{T}_{\boldsymbol{ij}\ldots\boldsymbol{kl}}\boldsymbol{e}_{\boldsymbol{i}}\boldsymbol{e}_{\boldsymbol{j}} \ldots \boldsymbol{e}_{\boldsymbol{k}}\boldsymbol{e}_l \tag{1.82}$$

where both the component and the base vector are functions of the vector diameter $\boldsymbol{r}$. In order to study its variation with a point, it is necessary to study the derivative of the nth tensor $\boldsymbol{T}$ to the sagittal $\boldsymbol{r}$. In the following studies, it is assumed that the partial derivative of the field function $\boldsymbol{T}$ to the coordinate x^l exists and is continuous.
Definition: The tensor field function $\boldsymbol{T}(\boldsymbol{r})$ is a finite differential of the delta $\boldsymbol{u}$ of $\boldsymbol{r}$

$$\boldsymbol{T}'(\boldsymbol{r};\boldsymbol{u}) = \lim_{\boldsymbol{h}\to 0}\frac{1}{\boldsymbol{h}}\,[\boldsymbol{T}(\boldsymbol{r}+\boldsymbol{h}\boldsymbol{u}) - \boldsymbol{T}(\boldsymbol{r})] \tag{1.83}$$

It can be proved that the finite differential is linear for the increment $\boldsymbol{u}$ of the sagittal diameter, so it is denoted as:

$$\boldsymbol{T}'(\boldsymbol{r};\boldsymbol{u}) = \boldsymbol{T}'(\boldsymbol{r}) \cdot \boldsymbol{u} \tag{1.84}$$

Where $\boldsymbol{T}'(\boldsymbol{r}) = d\boldsymbol{T}/d\boldsymbol{r}$ is called the derivative of the nth order tensor field function $\boldsymbol{T}(\boldsymbol{r})$ to the sagittal $\boldsymbol{r}$, which is $n + 1$ tensor according to the quotient of tensors.

It should be noted that for the tensor field function described in this chapter, the independent variable is the sagittal diameter $\boldsymbol{r}$, and $\boldsymbol{r}$ is a function of the curvilinear coordinate x^l (l = 1, 2,3):

$$\boldsymbol{r} = \boldsymbol{r}(\boldsymbol{x}^{\boldsymbol{l}}) \tag{1.85}$$

Thus, the tensor field function (the component of the tensor and the base vector) is also a function of coordinates, but the coordinate x^l is not a component of the vector $\boldsymbol{r}$.

In any curve coordinate system $\boldsymbol{r}$, the delta $\boldsymbol{u}$ of the vector diameter $\boldsymbol{r}$ at that point can be decomposed into

$$\boldsymbol{u} = \boldsymbol{u}_i \boldsymbol{e}_i \tag{1.86}$$

In order to further obtain the components of the derivative $\boldsymbol{T}(\boldsymbol{r})$ of the field function to the vector diameter in the coordinate system, the finite differentiation of the field function $\boldsymbol{T}(\boldsymbol{r})$ when the increment is the basis vector $\boldsymbol{e}_i$ is first studied:

$$\boldsymbol{T}'(\boldsymbol{r}; \boldsymbol{u}) = \lim_{\boldsymbol{h}\to 0} \frac{1}{\boldsymbol{h}} \left[\boldsymbol{T}(\boldsymbol{r} + \boldsymbol{h}\boldsymbol{u}) - \boldsymbol{T}(\boldsymbol{r})\right] \tag{1.87}$$

As shown in Figure 1.2, the sagittal diameter $\boldsymbol{r}$ has an incremental $h\boldsymbol{g}_i$, which is equivalent to the point where the field function is defined is moved from P’ (sagittal $\boldsymbol{r}$) to the point P’ (sagittal $\boldsymbol{r}$’) on the tangent of the coordinate line x^i:

$$\boldsymbol{r}' = \boldsymbol{r} + \boldsymbol{h}\boldsymbol{g}_{\boldsymbol{i}} \tag{1.88}$$

Its corresponding field function $\boldsymbol{T}(\boldsymbol{r})$ actually becomes a composite function of the coordinates $\boldsymbol{r}^l$ $(l = 1, 2, 3)$ by Eq. (1.85):

$$\boldsymbol{T} = \boldsymbol{T}\left(\boldsymbol{r}\left(\boldsymbol{x}^{\boldsymbol{l}}\right)\right) \tag{1.89}$$

Analogy with the definition of partial derivatives of multivariate functions, there are

$$\frac{\partial \boldsymbol{T}}{\partial \boldsymbol{x}^{\boldsymbol{i}}} = \lim_{\boldsymbol{h}\to 0} \frac{1}{\boldsymbol{h}} \left[\boldsymbol{T}\left(\boldsymbol{r}\left(\boldsymbol{x}^{\boldsymbol{l}} + \boldsymbol{h}\delta_{\boldsymbol{i}}^{\boldsymbol{l}}\right) - \boldsymbol{T}\left(\boldsymbol{r}\left(\boldsymbol{x}^{\boldsymbol{l}}\right)\right)\right)\right] (\boldsymbol{i} = 1, 2, 3) \tag{1.90}$$

In Eq. (1.90), l is neither a free nor a dumb indicator, but only indicates that T is a function of x_1, x_2, x_3, i is the indicator of the determined coordinate for which the partial derivative is obtained, and $i = 1$, 2, and 3 correspond to three formulas. The change of the sagittal diameter $\boldsymbol{r}\left(x^l\right)$ to $\boldsymbol{r}'' = \boldsymbol{r}\left(x^l h\delta_i^l\right)$, is equivalent to the point of the defined function in Figure 1.2 moving from P along the coordinate line x^i to the point P''. Comparing

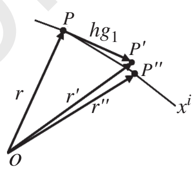

Figure 1.2 The change in the vector diameter in the tensor field function definition domain.

Eq. (1.87) with Eq. (1.90), it is noted that since the sagittal diameter at the P'' point can be expanded into a Taylor series:

$$\boldsymbol{r}\left(\boldsymbol{x}^l + \boldsymbol{h}\delta_i^l\right) = \boldsymbol{r}(\boldsymbol{x}^l) + \frac{\partial \boldsymbol{r}}{\partial \boldsymbol{x}^i}\boldsymbol{h} + \boldsymbol{o}(\boldsymbol{h}) = \boldsymbol{r}(\boldsymbol{x}^l) + \boldsymbol{h}\boldsymbol{g}_i + \boldsymbol{o}(\boldsymbol{h}) \tag{1.91}$$

Therefore the difference between the sagittal diameter $\boldsymbol{r}$ and $\boldsymbol{r}''$ is only a higher-order infinitesimal quantity, and therefore the difference between the values of the function at these two points is also a higher-order infinitesimal quantity, then the right-end limit of Eq. (1.87) and Eq. (1.90) must be equal, i.e.:

$$\boldsymbol{T}'(\boldsymbol{r}; \boldsymbol{g}_i) = \frac{\partial \boldsymbol{T}}{\partial \boldsymbol{x}^i} \tag{1.92}$$

According to the finite differentiation of the linear nature of the increment and Eq. (1.86) and Eq. (1.92), it can be seen:

$$\boldsymbol{T}'(\boldsymbol{r}\,; \boldsymbol{u}) = \boldsymbol{T}'\left(\boldsymbol{r}\,; \boldsymbol{u}^i\boldsymbol{g}_i\right) = \boldsymbol{u}^i\frac{\partial \boldsymbol{T}}{\partial \boldsymbol{x}^i} = \left(\frac{\partial \boldsymbol{T}}{\partial \boldsymbol{x}^i}\boldsymbol{g}^i\right)\boldsymbol{u} \tag{1.93}$$

Compare Eq. (1.93) with Eq. (1.84), since the increment is arbitrarily given:

$$\boldsymbol{T}'(\boldsymbol{r}) = \frac{\boldsymbol{dT}}{\boldsymbol{dr}} = \frac{\partial \boldsymbol{T}}{\partial \boldsymbol{x}^i}\boldsymbol{g}^i \tag{1.94}$$

It can be seen from the definition of finite differentiation and derivatives:

$$\boldsymbol{T}(\boldsymbol{r} + \boldsymbol{hu}) - \boldsymbol{T}(\boldsymbol{r}) = \boldsymbol{T}'(\boldsymbol{r}) + \boldsymbol{hu} + \boldsymbol{o}(\boldsymbol{h}) \tag{1.95}$$

Let

$$\boldsymbol{dr} = \boldsymbol{du} \tag{1.96}$$

then the main part of Eq. (1.95) is called the differentiation of the tensor field function, denoted as $\boldsymbol{dT}$, which satisfies the derivative:

$$\boldsymbol{dT} = \boldsymbol{T}(\boldsymbol{r})\cdot \boldsymbol{dr} \tag{1.97}$$

1.3.5 Gradient, Divergence, and Curl

1. **Gradient**

1. Gradient of Scalar Functions

   Let $f(\boldsymbol{r})$ be a scalar function of the sagittal diameter $\boldsymbol{r}$, then the differentiation of $f(\boldsymbol{r})$:

$$df = f(\boldsymbol{r} + d\boldsymbol{r}) - f(\boldsymbol{r}) = \frac{df}{d\boldsymbol{r}} \cdot d\boldsymbol{r} \tag{1.98}$$

According to the quotient rule of tensors, $\partial f/\partial \boldsymbol{r}$ is a vector and can be expressed as

$$\frac{\partial f}{d\boldsymbol{r}} = \frac{\partial f}{\partial\,(r_i)}\boldsymbol{e}_i \tag{1.99}$$

A gradient called the function $f(\boldsymbol{r})$ and is usually denoted by ∇f or grad f. ∇ is a vector operator

$$\nabla = \frac{\partial}{\partial \boldsymbol{r}_i}\boldsymbol{e}_i \tag{1.100}$$

Let $f_{,i} = \partial f/\partial \boldsymbol{r}$, where the subscript i indicates that the partial derivative of the coordinate component r_i is obtained, and therefore there is

$$\nabla f = f_{,i}\boldsymbol{e}_i \tag{1.101}$$

$$df = \nabla f \cdot d\boldsymbol{x} = (f_{,i}\boldsymbol{e}_i) \cdot (d\boldsymbol{x}_j\boldsymbol{e}_j) = f_{,i}dx_i \tag{1.102}$$

The geometric meaning of the gradient ∇f of the scalar function $f(\boldsymbol{r})$: $f(\boldsymbol{r}) = C$, which represents a surface in three-dimensional space, is called an equipotential surface, $f(\boldsymbol{r})$ is called a potential function, and the constant C takes different values and represents a different equipotential surface. Considering the neighbor point $\boldsymbol{r} + d\boldsymbol{r}$ at $\boldsymbol{r}$ on the equipotential surface, it is obvious that $f(\boldsymbol{r}) = C$, $f(\boldsymbol{r} + d\boldsymbol{r}) = C$, and the two equations are subtracted:

$$df = f(\boldsymbol{r} + d\boldsymbol{r}) - f(\boldsymbol{r}) = \nabla f \cdot d\boldsymbol{r} = 0 \tag{1.103}$$

Equation (1.103) shows that ∇f is orthogonal to $d\boldsymbol{r}$, i.e. the gradient ∇f of the scalar potential function $f(\boldsymbol{r})$, whose direction is orthogonal to the section of the equipotential surface, pointing to the direction in which the equipotential surface expands, which is the direction in which the potential function $f(\boldsymbol{r})$ changes the fastest.

Let $f(\boldsymbol{T})$ be a scalar function of the second-order tensor $\boldsymbol{T}$, then the differentiation of the function $f(\boldsymbol{T})$:

$$df = f(\boldsymbol{T} + d\boldsymbol{T}) - f(\boldsymbol{T}) = \frac{\partial f}{\partial \boldsymbol{T}} : d\boldsymbol{T} \tag{1.104}$$

From the quotient rule of tensors, $\frac{\partial f}{\partial \boldsymbol{T}}$ is a second-order tensor, which can be expressed as:

$$\frac{\partial f}{\partial \boldsymbol{T}} = \frac{\partial f}{\partial \boldsymbol{T}_{ij}}\boldsymbol{e}_i\boldsymbol{e}_j \tag{1.105}$$

$$df = \frac{\partial f}{\partial \boldsymbol{T}} : d\boldsymbol{T} = \frac{\partial f}{\partial \boldsymbol{T}_{ij}}d\boldsymbol{T}_{ij} \tag{1.106}$$

In a nine-dimensional Euclidean space consisting of the nine component tensors of the second-order tensor T_{ij}, $f(\boldsymbol{T}) = \mathrm{C}$ represents an equipotential surface, and the direction of the gradient $\partial f/\partial \boldsymbol{T}$ of the scalar potential function $f(\boldsymbol{T})$ is orthogonal to the "slice" of this equipotential surface, which is the direction in which the potential function $f(\boldsymbol{T})$ changes the fastest.

2. Gradient of Vector Functions

Let $\boldsymbol{V}(\boldsymbol{r})$ be a vector function of the vector diameter $\boldsymbol{r}$, then the differentiation of the function $\boldsymbol{V}(\boldsymbol{r})$ can be denoted as

$$d\boldsymbol{V} = \boldsymbol{V}(\boldsymbol{r} + d\boldsymbol{r}) - \boldsymbol{V}(\boldsymbol{r}) = \frac{\partial \boldsymbol{V}}{\partial \boldsymbol{r}} \cdot d\boldsymbol{r} \tag{1.107}$$

According to the quotient rule of tensors, $\partial V/\partial \boldsymbol{r}$ is a second-order tensor, called the gradient of the vector function $V(\boldsymbol{r})$, which can be represented by ∇V, and ∇ is still a vector operator:

$$\nabla \boldsymbol{V} = \frac{\partial \boldsymbol{V}}{\partial \boldsymbol{r}} = \frac{\partial \boldsymbol{V}}{\partial \boldsymbol{r}_j}\boldsymbol{e}_j = \frac{\partial V_i}{\partial \boldsymbol{r}_j}\boldsymbol{e}_i\boldsymbol{e}_j = V_{ij}\boldsymbol{e}_i\boldsymbol{e}_j \tag{1.108}$$

$$dV = \nabla V \cdot dx = V_{ij} x_j e_i \tag{1.109}$$

3. Gradient of Tensor Function
Defined by the gradient of a vector function, it can be generalized to a gradient of a tensor function. Let $\boldsymbol{T}(\boldsymbol{r})$ be the nth order tensor function of the vector $\boldsymbol{r}$, then the gradient ∇T of $\boldsymbol{T}(\boldsymbol{r})$ is an $n+1$ tensor

$$\nabla \boldsymbol{T} = \frac{\partial \boldsymbol{T}}{\partial \boldsymbol{r}} = \frac{\partial \boldsymbol{T}}{\partial \boldsymbol{r}_j} \boldsymbol{e}_j \tag{1.110}$$

Differentiation of the function $\boldsymbol{T}(\boldsymbol{r})$:

$$d\boldsymbol{T} = \boldsymbol{T}\,(r + d\boldsymbol{r}) - \boldsymbol{T}\,(\boldsymbol{r}) = \frac{\partial \boldsymbol{T}}{\partial \boldsymbol{r}} \cdot d\boldsymbol{r} \tag{1.111}$$

2. **Divergence**
For example, the divergence of the gradient ∇f of the scalar function $f(\boldsymbol{x})$:

$$\nabla \cdot \nabla f = \nabla^2 f = \frac{\partial \nabla f}{\partial \boldsymbol{r}_j} \cdot \boldsymbol{e}_j = f_{ij} \boldsymbol{e}_i \cdot \boldsymbol{e}_j = f_{ij} \tag{1.112}$$

Divergence of vector $\boldsymbol{V}$:

$$\nabla \cdot \boldsymbol{V} = \frac{\partial \boldsymbol{V}}{\partial \boldsymbol{r}_i} \cdot \boldsymbol{e}_j = \frac{\partial V_i}{\partial \boldsymbol{r}_i} \boldsymbol{e}_i \cdot \boldsymbol{e}_j = V_{i,j}. \tag{1.113}$$

The divergence of the tensor $\boldsymbol{T}$ is defined as $\nabla \cdot \boldsymbol{T}$:

$$\nabla \cdot \boldsymbol{T} = \frac{\partial \boldsymbol{T}}{\partial \boldsymbol{r}_j} \cdot \boldsymbol{e}_j \tag{1.114}$$

3. **Curl**
For example, the curl of the gradient ∇f of the scalar function: $f(\boldsymbol{r})$:

$$\nabla \times \nabla f = \frac{\partial \nabla f}{\partial \boldsymbol{r}_j} \times \boldsymbol{e}_j = f_{ij} \boldsymbol{e}_i \times \boldsymbol{e}_j = f_{ij} \boldsymbol{e}_{ijk} \boldsymbol{e}_k = 0 \tag{1.115}$$

Curl of vector $\boldsymbol{V}$:

$$\nabla \times \boldsymbol{V} = \frac{\partial \boldsymbol{V}}{\partial \boldsymbol{r}_j} \boldsymbol{e}_i \times \boldsymbol{e}_j = \boldsymbol{V}_{ij} \boldsymbol{e}_{ijk} \boldsymbol{e}_k \tag{1.116}$$

The curl of the tensor $\boldsymbol{T}$ is defined as $\nabla \times \boldsymbol{T}$:

$$\nabla \times \boldsymbol{T} = \frac{\partial \boldsymbol{T}}{\partial \boldsymbol{r}_i} \times \boldsymbol{e}_j \tag{1.117}$$

1.3.6 Green's Theorem and Stokes' Theorem

1. **Green's Theorem**
The Green transformation formula gives the transformation formula of the volume integral of the tensor function and the area integral of the closed domain. If an nth-order tensor field function T (n is any positive integer or zero) is defined on the volume

domain V in three-dimensional space, and if dv is the micro-unit volume on the volume domain and dS is the micro-unit surface area vector, then there is:

$$\int_{V} \nabla\cdot\boldsymbol{T}d\boldsymbol{V} = \int_{V} \boldsymbol{T}\cdot\boldsymbol{n}dS \tag{1.118}$$

where, the unit vector $\boldsymbol{n}$ represents the outer normal of dS. For example, if T is a second-order tensor, then there is:

$$\int_{V} T_{ij,j}e_i dV = \int_{V} T_{ij}n_j e_i dS \quad \text{or} \quad \int_{V} T_{ij,j}n_j dS = \int_{V} T_{ij}n_j dS \tag{1.119}$$

2. **Stokes' Theorem**

As shown in Figure 1.3, if the closed boundary of the open surface a is l, and the direction of l is right-handed with the direction n of the outer normal of da, then there is

$$\int_{a} d\boldsymbol{a}\cdot(\nabla\times\boldsymbol{v}) = \int_{l} ds\cdot\boldsymbol{v} \tag{1.120}$$

This is the Stokes formula. Generalizes $\boldsymbol{v}$ to tensors of any order φ:

$$\begin{cases} \int_{a} d\boldsymbol{a}\cdot(\nabla\times\boldsymbol{\varphi}) = \int_{l} d\boldsymbol{s}\cdot\boldsymbol{\varphi} \\ \int_{a} (\boldsymbol{\varphi}\times\nabla)\cdot d\boldsymbol{a} = -\int_{l} \boldsymbol{\varphi}\cdot d\boldsymbol{s} \end{cases} \tag{1.121}$$

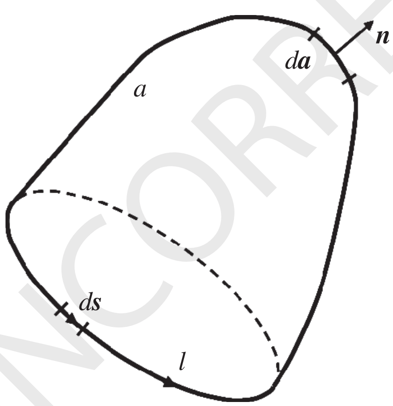

Figure 1.3 Open surface a and closed curve l.

1.4 Application of Tensors in Mechanics

Tensors, serving as both mathematical constructs and representations of physical quantities, play a foundational role in continuum mechanics. They provide the essential framework for expressing and interpreting physical laws in a manner that is independent of coordinate systems, thereby ensuring objectivity and consistency across different frames of reference. Within theoretical domains such as elasticity, plasticity, damage mechanics, and micromechanics, tensors are not merely symbolic tools; they are indispensable in the formulation of constitutive laws, the characterization of material anisotropy, and the description of mechanical and structural evolution.

In the theory of elasticity, second-order tensors are used to describe the internal state of stress and strain in materials, and to establish constitutive relations through stiffness or compliance tensors. In plasticity theory, it is critical to distinguish among the total strain tensor, the elastic strain tensor, and the plastic strain tensor, which together define the incremental response of materials. Yield criteria are typically constructed using tensor invariants, and consistent tangent operators—expressed as fourth-order tensors—are employed in the numerical integration of elastoplastic constitutive models.

In damage mechanics, tensor-valued internal variables are introduced to describe the degradation of material stiffness. These damage tensors modify the stress-strain relationship, leading to the concept of effective stress, which underpins the formulation of damage-coupled constitutive laws.

At the microscale, tensors are also used to characterize crystal orientation, lattice distortion, and slip systems in polycrystalline materials. In micromechanics, the deformation gradient tensor serves as a key mathematical tool for establishing multiscale relations between microstructural behavior and macroscopic responses.

Table 1.1 summarizes representative tensors frequently used in mechanics and indicates their respective tensor orders, reflecting their structural and physical roles across various scales and formulations.

Table 1.1 Some commonly used tensors in mechanics.

Tensor type	Order	Application
Stress/strain tensor	2	Continuum mechanics, structural mechanics
Inertia tensor	2	Rigid body dynamics, rotor mechanics
Flexibility/stiffness tensor	4	Constitutive relations, material modeling
Damage tensor	2	Fracture mechanics, material degradation, fatigue
Deformation gradient tensor	2	Large deformation theory, nonlinear mechanics

1.5 Fundamental Laws of Continuum Mechanics

The subject of continuum mechanics is the mechanical behavior of continuous media—systems or materials that satisfy the laws of Newtonian mechanics in three-dimensional Euclidean space under the assumption of continuous time. Specifically, it addresses the macroscopic mechanical properties of matter treated as a continuum. As a discipline, continuum mechanics is broadly categorized into solid mechanics, fluid mechanics, and rheology. Solid mechanics can further be subdivided into elastoplastic mechanics and viscoelastoplastic mechanics based on material response characteristics.

Rational mechanics, also known as mathematical mechanics, provides the theoretical foundation for continuum mechanics. It emphasizes the use of rigorous logical reasoning and abstract mathematical formulations to study mechanical systems in a unified and systematic way. D'Alembert, in his 1743 treatise Dynamics, asserted that rational mechanics, like geometry, must be founded upon axioms that are self-evident and universally valid [11]. A core objective of rational mechanics is to establish a set of axioms and general principles that are applicable to all types of continua.

In the context of scientific methodology, axioms are fundamental truths distilled from extensive empirical experience. They are considered self-evident, irreducible, and beyond the need for proof. Axioms serve as starting points for deriving more complex theories. Euclid's seminal work Elements exemplifies the axiomatic approach: all its theorems are logically deduced from a finite set of axioms. This method has since become a hallmark of rigorous scientific reasoning and has influenced the development of other knowledge systems.

The deformation and motion of a continuum must obey fundamental physical laws, which are commonly expressed in the form of conservation equations. During deformation, changes in physical quantities arise from internal sources and external influences. Typically, internal sources are represented by volume integrals, while external influences are modeled via surface integrals. The fundamental conservation laws of continuum mechanics include:

a. Conservation of mass,
b. Conservation of linear momentum,
c. Conservation of angular momentum, and
d. Conservation of energy.

When thermodynamic effects are considered, the Clausius–Duhem inequality, representing the second law of thermodynamics, must also be satisfied. Collectively, these laws constitute the field equations of continuum mechanics and form the core theoretical framework for modeling physical behavior in solids and fluids.

1.5.1 Law of Conservation of Mass

In the process of deformation and movement, the total mass of the continuum remains unchanged, and the total mass of the same continuum remains unchanged for any selected configuration. Mass is a metric that every substance possesses, and is both nonnegative and additive. If the mass is absolutely continuous with respect to spatial coordinates, there is a mass density $\rho\,(\boldsymbol{x},\ t)$. If the volume of the continuum is v, its mass is:

$$m = \int_v \rho\,(\boldsymbol{x},\ t)\ \mathrm{d}v \tag{1.122}$$

The law of conservation of mass can be expressed as:

$$\int_v \rho\,(\boldsymbol{x},\ t)\ \mathrm{d}v = \int_{V_0} \rho_0\,(\boldsymbol{x},\ t)\ \mathrm{d}V \tag{1.123}$$

where, V_0 represents the volume occupied by the continuum in the reference configuration; ρ_0 is the mass density in the reference configuration.

Since the volume change of the two configurations is equal to the value of the deformation gradient determinant, Eq. (1.123) can be expressed as:

$$\int_{V_0} (\rho_0 - \rho J)\ \mathrm{d}V = 0 \tag{1.124}$$

Equation (1.124) is the continuity equation in the form of Lagrange integral, which requires that it is valid for any volume element, then:

$$\rho_0 = \rho J \tag{1.125}$$

Since ρ_0 is a constant, both sides of Eq. (1.125) take the derivative of time at the same time, there is:

$$\frac{\mathrm{d}}{\mathrm{d}t}\,(\rho J) = J\frac{\mathrm{d}\rho}{\mathrm{d}t} + \rho\frac{\mathrm{d}J}{\mathrm{d}t} = J\frac{\mathrm{d}\rho}{\mathrm{d}t} + \rho J \mathrm{div}\boldsymbol{v} = 0 \tag{1.126}$$

thus obtain:

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} + \rho \mathrm{div}\boldsymbol{v} = 0 \tag{1.127}$$

Equation (1.127) is the differential form of the continuity equation, also known as the Euler continuity equation.

1.5.2 Law of Conservation of Momentum

At a certain time t, the volume of the continuum is v, and the outer surface area is s. The physical force of the continuum is $\boldsymbol{b}$, the surface force is $\boldsymbol{t}_n$, $\boldsymbol{n}$, which is the unit vector of the microelement's out-of-plane normal, and there is a velocity field $\boldsymbol{v}$ in the region. According to the definition of momentum, there is

$$\boldsymbol{M} = \int_v \rho\boldsymbol{v}\mathrm{d}v \tag{1.128}$$

If the resultant force acting on the volume v is recorded as $\boldsymbol{F}$, it can be obtained according to the force acting on the continuum.

$$\boldsymbol{F} = \oint_s \boldsymbol{t}_n \mathrm{d}s + \int_v \rho\boldsymbol{b}\mathrm{d}v \tag{1.129}$$

Since the rate of change of momentum with respect to time is equal to the net force acting on the object, we can get from Eq. (1.128)

$$\frac{d\boldsymbol{M}}{dt} = \boldsymbol{F} = \frac{d}{dt}\int_v \rho \boldsymbol{v} dv = \oint_s \boldsymbol{t}_n ds + \int_v \rho \boldsymbol{b} dv \tag{1.130}$$

Combining Eq. (1.122), there is

$$\frac{d}{dt}\int_v \rho \boldsymbol{v} dv = \int_v \rho \frac{d\boldsymbol{v}}{dt} dv = \oint_s \boldsymbol{\sigma n} ds + \int_v \rho \boldsymbol{b} dv \tag{1.131}$$

According to Gauss's theorem, the area integral of Eq. (1.131) is converted into a volume integral:

$$\int_v \rho \frac{d\boldsymbol{v}}{dt} dv = \int_v \mathrm{div}\boldsymbol{\sigma} dv + \int_v \rho \boldsymbol{b} dv \tag{1.132}$$

Eq. (1.132) is the law of conservation of momentum in integral form. Since the acceleration $\boldsymbol{a}(\boldsymbol{x}, t) = d\boldsymbol{v}/dt$, there is

$$\mathrm{div}\boldsymbol{\sigma} + \rho \boldsymbol{b} = \rho \boldsymbol{a} \tag{1.133}$$

Eq. (1.133) is the Cauchy momentum equation, also known as the equation of motion. Where div is the divergence operator in the current configuration.

1.5.3 Law of Conservation of Moment of Momentum

The moment of momentum, also known as angular momentum, is the moment of momentum with respect to a point O. If the volume of the element is v and $\boldsymbol{x}$ is the position vector of the element in the coordinate system, then the moment of momentum of the element with respect to the coordinate origin O is

$$\boldsymbol{L} = \int_v (\boldsymbol{x} \times \boldsymbol{v}) \rho dv \tag{1.134}$$

Or

$$L_i = \int_v e_{ijk} x_j v_k \rho dv \tag{1.135}$$

The rate of change with time of the moment of momentum of any part of the continuum with respect to a point is equal to the combined moment of the forces acting on the continuum, including physical and surface forces. Therefore, the integral form of the law of conservation of moment of momentum is

$$\int_s e_{ijk} x_j t_k^{(n)} ds + \int_v e_{ijk} x_j b_k \rho dv = \frac{d}{dt}\int_v e_{ijk} x_j v_k \rho dv \tag{1.136}$$

1.5.4 Law of Conservation of Energy

At any time t, the kinetic energy of a substance of volume v in the current configuration is

$$K = \frac{1}{2}\int_v \rho \boldsymbol{v} \cdot \boldsymbol{v} dv \tag{1.137}$$

Take the first derivative of time on both sides of Eq. (1.137):

$$\dot{K} = \frac{1}{2}\int_v \frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{v}\cdot\boldsymbol{v})\,\rho \mathrm{d}v = \int_v \boldsymbol{v}\cdot\rho\boldsymbol{a}\mathrm{d}v \tag{1.138}$$

By replacing the motion equation of Eq. (1.133) into Eq. (1.138) and using the formula$(\boldsymbol{v}\boldsymbol{\sigma})\,\mathrm{div} = \boldsymbol{v}\,(\boldsymbol{\sigma}\mathrm{div}) + (\boldsymbol{v}\otimes\nabla) : \boldsymbol{\sigma}$, Eq. (1.138) can finally be written in the following integral form:

$$\dot{K} + \int_v \boldsymbol{\sigma} : \boldsymbol{D}\mathrm{d}v = \oint_s \boldsymbol{v}\cdot\boldsymbol{t}_{(n)}\mathrm{d}s + \int_v \rho\boldsymbol{v}\cdot\boldsymbol{b}\mathrm{d}v \tag{1.139}$$

The second term on the left of Eq. (1.139) represents the rate of change of strain rate with respect to time, denoted as $\mathrm{d}U/\mathrm{d}t$. The right side of the equation is the power of the surface force and the physical force.

If only mechanical energy and heat energy are considered, and other energies, such as chemical energy and electromagnetic energy, are not considered, the law of conservation of energy is the first law of thermodynamics. Therefore, the first law of thermodynamics can be understood as a narrow form of conservation of energy.

If the internal energy per unit mass is denoted as u, the total internal energy of the substance

$$U = \int_v \rho u \mathrm{d}v \tag{1.140}$$

The total energy P of matter is the sum of kinetic energy and internal energy, that is, $P = K + U$.

According to the first law of thermodynamics, the first derivative of total energy with respect to time is equal to the power of the external force acting on the volume and the heat applied externally to the volume per unit time. So, the first derivative of the total energy with respect to time has

$$\dot{U} = \int_v \rho\dot{u}\mathrm{d}v = \oint_s \boldsymbol{v}\cdot\boldsymbol{t}_{(n)}\mathrm{d}s + \int_v \rho\boldsymbol{v}\cdot\boldsymbol{b}\mathrm{d}v - \int_s \boldsymbol{q}\cdot\boldsymbol{n}\mathrm{d}s + \int_v \rho\lambda\mathrm{d}v \tag{1.141}$$

where, $\boldsymbol{q}$ is the heat flow vector, heat flux, per unit time per unit area; λ is the radiation heat obtained per unit time per unit mass; $\boldsymbol{n}$ is the unit external normal vector of micro-plane ds. The heat flow is specified to be positive in and negative out.

According to the kinetic energy theorem Eq. (1.139), it can be obtained that the rate of internal energy with respect to time is

$$\dot{U} = \int_v \rho\dot{u}\mathrm{d}v = -\int_s \boldsymbol{q}\cdot\boldsymbol{n}\mathrm{d}s + \int_v \rho\lambda\mathrm{d}v + \int_v \boldsymbol{\sigma} : \boldsymbol{D}\mathrm{d}v \tag{1.142}$$

Equation (1.142) is the first law of thermodynamics in integral form.

According to Gauss's theorem, converting the area integral of Eq. (1.142) into a volume integral gives the first law of thermodynamics in differential form:

$$\rho\dot{u} = -div\boldsymbol{q} + \rho\lambda + \boldsymbol{\sigma} : \boldsymbol{D} \tag{1.143}$$

Equation (1.143) is the energy conservation equation of the current configuration, also known as the local energy equation, which shows that the rate of change of internal energy with time is equal to the sum of the stress power and the heat supplied to the medium.

1.5.5 Second Law of Thermodynamics

The first law of thermodynamics establishes the conservation and conversion relationship between mechanical and thermal energy but does not account for the irreversibility observed in many real-world processes. In nature, most physical and mechanical transformations are inherently irreversible, necessitating a more restrictive principle to describe them.

The second law of thermodynamics addresses this limitation by introducing the concept of entropy and providing criteria for determining whether a process is reversible. In the context of continuum mechanics, the second law serves as a fundamental constraint to assess the thermodynamic admissibility of deformation and heat transfer processes. It can be expressed in various equivalent forms, one of which states that: it is impossible for heat to spontaneously transfer from a colder body to a hotter body without the involvement of external work or additional effects. To establish a mathematical description of this law, two state functions, thermodynamic temperature T and entropy S of the system, are used. The total entropy of the system is equal to the sum of the entropy of each part. In continuum mechanics, specific entropy, entropy of per unit mass, or entropy density s is commonly used, so total entropy $S = \int_{\upsilon} \rho s \mathrm{d}\upsilon$. Changes in system entropy are affected by changes in the surrounding environment and within the system

$$\mathrm{d}s = \mathrm{d}s^e + \mathrm{d}s^i \tag{1.144}$$

where, $\mathrm{d}s$ is the amount of change in specific entropy; $\mathrm{d}s^e$ is the amount of change caused by external action; $\mathrm{d}s^i$ is the increment caused by internal change. $\mathrm{d}s^i$ is not negative, for reversible processes, it equals to zero; for irreversible processes, $\mathrm{d}s^i$ is positive.

According to the second law of thermodynamics, in a continuum of volume υ, the rate of change of total entropy with time is no less than the sum of the entropy flowing through the surface of the continuum and the entropy generated by the internal volume sources. This principle is expressed in integral form as Clausius inequality:

$$\dot{S} = \frac{\mathrm{d}}{\mathrm{d}t} \int_{\upsilon} \rho s \mathrm{d}\upsilon \geq \int_{\upsilon} \frac{\lambda}{T} \rho \mathrm{d}\upsilon - \int_{s} \frac{1}{T} \boldsymbol{q} \cdot \boldsymbol{n} \mathrm{d}s \tag{1.145}$$

For reversible process, Eq. (1.145) takes equal sign; For irreversible processes, take a greater than sign.

The difference between the two ends of the inequality is entropy generation Eq. (1.145) rate Γ in the volume υ, which represents the entropy generation rate per unit mass by γ, that is

$$\Gamma = \int_{\upsilon} \rho \gamma \mathrm{d}\upsilon, \gamma \geq 0 \tag{1.146}$$

From Eq. (1.145) and Eq. (1.146) written as the entropy balance equation in integral form:

$$\int_{\upsilon} \rho \dot{s} \mathrm{d}\upsilon = \int_{\upsilon} \frac{\lambda}{T} \rho \mathrm{d}\upsilon - \int_{s} \frac{1}{T} \boldsymbol{q} \cdot \boldsymbol{n} \mathrm{d}s + \int_{\upsilon} \rho \gamma \mathrm{d}\upsilon \tag{1.147}$$

By using Gauss theorem, the aspect integral of Eq. (1.148) is transformed into volume integral, and the entropy balance equation in differential form can be obtained as follows:

$$\rho \dot{s} = \rho \left(\frac{\lambda}{T} + \gamma \right) - \mathrm{div} \left(\frac{\boldsymbol{q}}{T} \right) \tag{1.148}$$

1.6 Constitutive Model Construction Principle

The constitutive relation in continuum mechanics refers to the fundamental relationship between stress and strain within a given material. While the foundational principles of continuum mechanics—namely the conservation of mass, momentum, and energy, along with the second law of thermodynamics (principle of entropy increase)—govern the general behavior of all continua, they are insufficient to uniquely determine the mechanical response of specific materials. Due to variations in material type, microstructure, and internal mechanisms, different substances may exhibit significantly different responses under identical external stimuli. To capture these unique behaviors, it is essential to develop constitutive equations tailored to specific materials. These equations enable the differentiation of materials within the continuum framework and provide a precise description of deformation and motion for particular substances.

The formal establishment of constitutive axioms began in the 1950s. In 1950, Oldroyd emphasized that rheological constitutive relations must possess proper invariance properties [12]. Subsequently, Noll introduced the three constitutive principles for continuum mechanics: the principle of determinacy (stress certainty), the local action principle, and the principle of objectivity (or frame-indifference) [13]. Later, researchers such as Wang and Truesdell and Eringen expanded this foundation [14, 15]. Eringen ultimately proposed eight constitutive axioms:

- **a.** Axiom of Causality
- **b.** Axiom of Determinacy (Certainty)
- **c.** Axiom of Equal Existence
- **d.** Axiom of Objectivity (Frame-Indifference)
- **e.** Axiom of Material Invariance
- **f.** Axiom of Domain
- **g.** Axiom of Memory
- **h.** Axiom of Compatibility

Each of these axioms plays a critical role in constraining and formulating constitutive models that are physically meaningful and mathematically consistent. A detailed explanation of each axiom is provided below.

1. **Causality Axiom**
   In continuum thermomechanics, the motion, temperature, and electric charge of a material point are typically treated as independent constitutive variables, while quantities such as internal energy density, entropy density, stress tensor, and heat flux vector are regarded as dependent constitutive variables. These dependent variables are determined by the evolution and interaction of the independent variables and represent the material's thermomechanical response.

When analyzing thermodynamic processes in a continuum without considering the coupling effects of mechanical deformation, electromagnetic fields, or chemical reactions, the set of independent constitutive variables is simplified to include only the motion $\mathbf{x}$ and temperature T. These variables are functions of both the material position vector $\boldsymbol{X}$ and time t, and can be expressed as:

$$\boldsymbol{x} = \boldsymbol{x}(\boldsymbol{X}, t), \quad T = T(\boldsymbol{X}, t) \tag{1.149}$$

When the independent variable is determined, other dependent variables are also determined, such as velocity vector, deformation gradient, etc., which can be obtained from the continuity equation.

2. **Axiom of Certainty**
The thermodynamic constitutive functional and stress state of a point of matter at a given time are determined only by the motion and temperature history of all points in the matter, and have nothing to do with future motions. This axiom excludes the influence of any other factors outside the object and in the future on the properties of the material point X. As long as all the past motions and temperature history of the object are known, the present behavior of the object is completely determined. From the axiom of certainty, we know that the stress $\boldsymbol{\sigma}$ at the point X of a substance is completely determined by the entire history of the motion of the object up to the present time.
3. **Axiom of Equal Existence**
Initially, all constitutive functionals should be expressed in terms of the same set of independent constitutive variables, and no variable should be arbitrarily excluded. The exclusion of any variable must be justified by contradictions that arise from its inclusion, either theoretically or experimentally.
Eringen emphasized that this axiom requires treating all variables on an equal footing—no class of variable may be disregarded or rejected without sufficient justification [15]. Only when restrictions are imposed by fundamental principles—such as the axiom of objectivity, the axiom of compatibility, or material symmetry considerations—can the admissibility of certain variables be limited.
4. **Axiom of Objectivity**
The mechanical behavior of materials must be independent of the choice of reference frame. In other words, a constitutive model should remain invariant under changes in the coordinate system or reference configuration. This requirement is formalized in the axiom of objectivity (or frame-indifference), which states that the constitutive relations must be invariant under superposed rigid body motions.
Consequently, all physical quantities appearing in constitutive equations must be objective quantities—their values and physical meaning must not change under transformations such as rotation or translation of the reference frame. For two physically identical processes observed from different frames, the constitutive functional must remain the same.
When formulated using tensor notation, which inherently respects transformation laws and objectivity, this axiom is naturally satisfied. Thus, the adoption of tensorial descriptions is essential for ensuring the physical consistency and frame-independence of constitutive models in continuum mechanics.

5. **Axiom of Invariance of Matter**
The constitutive equation must faithfully reflect the material symmetry of the substance. The mechanical response of materials is significantly influenced by their microstructural characteristics, which often give rise to specific symmetry properties. For instance, the crystallographic orientation in crystalline solids can lead to anisotropic behavior, where material responses vary depending on the direction of loading.
These inherent symmetries impose constraints on permissible forms of the constitutive relations by restricting the functional dependence of material response on deformation or motion. Accordingly, the constitutive equations must be formulated in a manner that is consistent with the symmetry group of the material.
It is important to note that different physical properties of the same material may exhibit different symmetry characteristics. For example, a material may be mechanically isotropic—exhibiting the same mechanical response in all directions—yet electrically anisotropic due to directional variations in its conductive properties. Therefore, when constructing constitutive models for coupled physical fields (e.g. thermo-electro-mechanical systems), one must carefully account for the specific symmetry properties relevant to each physical phenomenon.
6. **Axiom of Domain**
The motion of a material point within a body is generally influenced by the motion of surrounding points. Among these, the influence is strongest from those within its local neighborhood, often referred to as the material point's field of influence. As the distance increases, the effect of other points on the motion of the given point typically decays rapidly and becomes negligible.
From the perspective of microscopic or molecular mechanics, the deformation behavior of a material point is ultimately governed by interatomic and intermolecular interactions. However, such interactions decay exponentially with distance, often becoming insignificant beyond approximately 10 interatomic spacings. This physical observation provides the basis for the local action axiom in continuum mechanics, which states that the stress at a material point is determined solely by the state of neighboring points within a finite, localized region, and is independent of the state of points that lie beyond this region.
Consequently, constitutive models are commonly formulated under the assumption of locality, wherein stress is a function of local strain and its derivatives, excluding the need for long-range interactions in standard continuum formulations.
7. **Axiom of Memory**
The influence of past constitutive variables on the present state diminishes with increasing temporal distance. This axiom, often referred to as the memory fading principle, reflects the intrinsic history-dependence of material behavior: the current state of a material (e.g. stress, strain, internal energy) is affected by its deformation history. However, the influence of past states decays over time, and values of constitutive variables from the distant past eventually become negligible.
In essence, this axiom captures the temporal nonlocality in materials with memory (e.g. viscoelastic materials), while asserting that the weighting of historical effects decreases as one move further away from the present. Mathematically, this often

leads to formulations where historical effects are incorporated through convolution integrals with decaying kernel functions, ensuring that long-past events exert progressively less influence on the current constitutive response.

8. **Consistency Axiom**
The constitutive relation must be consistent with the fundamental framework of continuum mechanics, including the geometrical relations, equilibrium equations, and the governing laws of thermodynamics. Specifically, a valid constitutive model must satisfy all the essential conservation laws—namely, the conservation of mass, linear momentum, angular momentum, and energy—as well as the Clausius–Duhem inequality, which expresses the second law of thermodynamics in differential form.

This requirement ensures that the constitutive formulation does not contradict the foundational physical principles governing the behavior of continuous media. Accordingly, this principle is referred to as the axiom of consistency, underscoring the necessity for constitutive equations to align with both mechanical and thermodynamic admissibility.

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